

Anomalous scaling, nonlocality, and anisotropy in a model of the passively advected vector field

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(Received 5 April 2001; published 26 September 2001)

A model of the passive vector quantity advected by the Gaussian velocity field with the covariance $\propto \delta(t-t')|\mathbf{x}-\mathbf{x}'|^\varepsilon$ is studied; the effects of pressure and large-scale anisotropy are discussed. The inertial-range behavior of the pair correlation function is described by an infinite family of scaling exponents, which satisfy exact *transcendental* equations derived explicitly in d dimensions by means of the functional techniques. The exponents are organized in a hierarchical order according to their degree of anisotropy, with the spectrum unbounded from above and the leading (minimal) exponent coming from the isotropic sector. This picture extends to higher-order correlation functions. Like in the scalar model, the second-order structure function appears nonanomalous and is described by the simple dimensional exponent: $S_2 \propto r^{2-\varepsilon}$. For the higher-order structure functions, $S_{2n} \propto r^{n(2-\varepsilon)+\Delta_n}$, the anomalous scaling behavior is established as a consequence of the existence in the corresponding operator product expansions of “dangerous” composite operators, whose *negative* critical dimensions determine the anomalous exponents $\Delta_n < 0$. A close formal resemblance of the model with the stirred Navier-Stokes equation reveals itself in the *mixing* of relevant operators and is the main motivation of the paper. Using the renormalization group, the anomalous exponents are calculated in the $O(\varepsilon)$ approximation, in large d dimensions, for the even structure functions up to the twelfth order.

DOI: 10.1103/PhysRevE.64.046310

PACS number(s): 47.27.-i, 47.10.+g, 05.10.Cc

I. INTRODUCTION

The investigation of intermittency and anomalous scaling in fully developed turbulence remains essentially an open theoretical problem. Much effort has been invested recently into the understanding of the inertial-range behavior of the passive scalar. Both the real experiments and numerical simulations suggest that the breakdown of the classical Kolmogorov-Obukhov theory [1] is even more strongly pronounced for a passively advected scalar field than for the turbulent velocity itself. On the other hand, the problem of passive advection appears easier tractable theoretically; see Ref. [2] and references therein.

The most progress has been achieved for the so-called rapid-change model of the passive scalar advection by a self-similar white-in-time velocity field [3]. The model is interesting because of the insight it offers into the origin of intermittency and anomalous scaling in turbulence: anomalous exponents have been calculated on the basis of a microscopic model and within controlled approximations [4–7]. Within the “zero-mode approach” to the rapid-change model, proposed in Refs. [4–6], nontrivial anomalous exponents are related to the zero modes (homogeneous solutions) of the closed exact differential equations satisfied by the equal-time correlations. In this sense, the model appears exactly solvable. A recent review and more references can be found in Ref. [2].

In Ref. [8] and subsequent papers [9–14], the field theoretic renormalization group (RG) and the operator product expansion (OPE) were applied to the model [3–5]. The feature specific to the theory of turbulence is the existence in the corresponding field theoretical models of the composite operators with *negative* scaling (critical) dimensions. Such operators, termed “dangerous” in [8–14], give rise to anomalous scaling, i.e., the singular dependence on the infrared (IR) scale with certain nonlinear anomalous exponents.

The OPE and the concept of dangerous operators for the Navier-Stokes (NS) turbulence were introduced in Ref. [15]; detailed review and bibliography can be found in [16,17]. The relationship between the anomalous exponents and dimensions of composite operators was anticipated in Ref. [18] for the stochastic hydrodynamics and in [4,5,19] for the Kraichnan model within certain phenomenological formulation of the OPE, the so-called “additive fusion rules,” typical to the models with multifractal behavior [20]. A similar picture naturally arises within the context of the Burgers turbulence and growth phenomena [21,22].

Important advantages of the RG approach are its universality and calculational efficiency: a regular systematic perturbation expansion for the anomalous exponents was constructed, similar to the well-known ε expansion in the theory of phase transitions, and the exponents were calculated in the second [8–11] and third [12] orders of that expansion. For passively advected vector fields, any calculation of the exponents for higher-order correlations calls for the RG techniques already in the $O(\varepsilon)$ approximation [9,23,24]. Furthermore, the RG approach is not related to the aforementioned solvability of the rapid-change model and can also be applied to the case of finite correlation time or non-Gaussian advecting field [14].

Recent research on the Kraichnan model and its descendants has mostly been concentrated on the passive scalar advection. The large-scale transport of vector quantities exhibits more interesting behavior; see monograph [1] and references therein. In this paper, we study the anomalous scaling and effects of anisotropy and pressure, for the passive *vector* field advected by the rapid-change velocity field. The model has already been introduced and discussed independently in Refs. [25] and [26].

Before explaining our motivations, which follow the same lines as those of Refs. [25,26], we shall discuss the definition of the model in detail.

We shall confine ourselves to the case of transverse (solenoidal) passive $\theta(x) \equiv \{\theta_i(t, \mathbf{x})\}$ and advecting $\mathbf{v}(x) \equiv \{v_i(t, \mathbf{x})\}$ vector fields and the advection-diffusion equation of the form

$$\nabla_i \theta_i + \partial_t \mathcal{P} = \nu_0 \Delta \theta_i + f_i, \quad \nabla_i \equiv \partial_i + (v_j \partial_j), \quad (1.1)$$

where $\mathcal{P}(x)$ is the pressure, ν_0 is the diffusivity, Δ is the Laplace operator, and $f_i(x)$ is a transverse Gaussian stirring force with zero mean and covariance

$$\langle f_i(x) f_j(x') \rangle = \delta(t - t') C_{ij}(\mathbf{r}/L), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}'. \quad (1.2)$$

The parameter L is an integral scale related to the stirring and C_{ij} is a dimensionless function finite as $L \rightarrow \infty$. Its precise form is not essential; for generality, it is not assumed to be isotropic. Therefore, the force maintains the steady state and is also a source of the large-scale anisotropy in the system.

The velocity $\mathbf{v}(x)$ obeys a Gaussian distribution with zero mean and covariance

$$\langle v_i(x) v_j(x') \rangle = D_0 \delta(t - t') \int \mathcal{D}\mathbf{p} P_{ij}(\mathbf{p}) p^{-d-\varepsilon} \exp[i(\mathbf{p}\mathbf{r})]. \quad (1.3)$$

Here and below \mathbf{p} is the momentum, $p = |\mathbf{p}|$, $\mathcal{D}\mathbf{p} = d\mathbf{p}/(2\pi)^d$, $P_{ij}(\mathbf{p}) \equiv \delta_{ij} - p_i p_j / p^2$ is the transverse projector, $D_0 > 0$ is an amplitude factor and d is the dimensionality of the \mathbf{x} space. The exponent $0 < \varepsilon < 2$ plays in the RG approach the same role as the parameter $\varepsilon = 4 - d$ does in the RG theory of critical behavior. The IR regularization is provided by the cutoff in integral (1.3) from below at $p = m$, where $1/m$ is another integral scale; the precise form of the cutoff is not essential. In what follows, we shall not distinguish the two IR scales, setting $m \sim 1/L$. The relations

$$D_0 / \nu_0 = g_0 = \Lambda^\varepsilon \quad (1.4)$$

define the coupling constant g_0 (i.e., the formal expansion parameter in the ordinary perturbation theory) and the characteristic ultraviolet (UV) momentum scale Λ .

Due to the transversality conditions, $\partial_i \theta_i = \partial_i v_i = 0$, the pressure can be expressed as the solution of the Poisson equation,

$$\Delta \mathcal{P} = -\partial_i v_j \partial_j \theta_i. \quad (1.5)$$

The issue of interest is, in particular, the behavior of the equal-time structure functions

$$S_n(\mathbf{r}) \equiv \langle [\theta_r(t, \mathbf{x}) - \theta_r(t, \mathbf{x}')]^n \rangle \quad (1.6)$$

in the inertial range, specified by the inequalities $1/\Lambda \ll r \ll L \sim 1/m$. Here $\theta_r = \theta_i r_i / r$ is the component of the passive field along the direction $\mathbf{r} = \mathbf{x} - \mathbf{x}'$, an analog of the streamwise component of the turbulent velocity field in real experiments.

The general symmetry of the vector problem permits one to add on the left-hand side of the advection-diffusion equation the ‘‘stretching term’’ of the form $(\theta_j \partial_j) v_i$. This general vector model is studied in Ref. [24], and its special magnetic

case, where the pressure term disappears, was studied earlier in a number of papers in detail; see Refs. [23,27–30].

From the physics viewpoints, the model (1.1)–(1.3) can be considered as the linearized NS equation with the prescribed statistics of the background field \mathbf{v} and the additional convention that the field \mathbf{v} is ‘‘soft’’ and the perturbation θ is ‘‘hard,’’ that is, $\partial \theta \gg \partial \mathbf{v}$. Our motivation, however, is different: a close formal resemblance of the model (1.1)–(1.3) with the stirred NS turbulence.

Although the model (1.1) is formally a special case of the general ‘‘A model’’ [24], it appears in a sense exceptional and requires special investigation. In this case, the stretching term is absent and the analog of the kinetic energy $\theta^2(x)$ is conserved, as for the passive scalar and stochastic NS equations. An important consequence is the existence of a constant flux solution, characteristic of the real NS turbulence [1]. The second-order structure function appears nonanomalous and is described by the simple ‘‘dimensional’’ exponent, $S_2 \propto r^{2-\varepsilon}$.

Furthermore, since only *derivatives* of the field θ enter into Eq. (1.1), the latter possesses additional symmetry $\theta \rightarrow \theta + \text{const}$. The leading anomalous exponents in the magnetic and general cases are determined by the composite operators built of the field θ without derivatives [23,24]; in our case they become trivial and the leading nontrivial exponents are related to the composite operators built solely of the *gradients* of θ .

A similar distinction exists also between the density and tracer scalar fields advected by a compressible velocity, as discussed in Ref. [9] in detail. But in contrast with the scalar case, where the leading exponent for a given-order structure function is determined by an individual composite operator even in the presence of the $\theta \rightarrow \theta + \text{const}$ invariance [8–10], in the vector model (1.1) the inertial-range behavior of any structure function is determined by a *family* of composite operators with the same symmetry and dimension, and in order to find the corresponding *set* of exponents and to identify the leading contribution, one has to consider the renormalization of the whole family, which implies the *mixing* of individual operators.

In the scalar case, the anomalous exponents for all structure functions are given by a single expression that includes n , the order of a function, as a parameter [4,5,8]. This remains true for the vector models with the stretching term [23,24]. In the special vector model (1.1), the number and the form of the operators entering into the relevant family depend essentially on n , and different structure functions should be studied separately. As a result, no general expression valid for all n exists in the model, in contrast with the scalar [4,5], magnetic [23], and general vector [24] models.

In this respect, the model (1.1) is one step closer to the nonlinear NS equation, where the inertial-range behavior of structure functions is believed to be related with the Galilean-invariant (and hence built of the velocity gradients) operators, which mix heavily in renormalization; see [16,17] and references therein.

Another important question recently addressed is the effects of large-scale anisotropy on the inertial-range statistics; see, e.g., Ref. [2] and references therein. In particular, it was

shown that in the presence of anisotropic forcing, the exponents describing the inertial-range scaling of the passively advected scalar [14,31] and vector [23,29,30] fields and the turbulent velocity field itself [32–35] are organized in a hierarchical order according to their degree of anisotropy, with the leading contribution coming from the isotropic sector. The consistency of this picture with the presence of nonlocal terms in the equations for the correlation functions, caused by the pressure contributions, has been addressed recently [36] and answered positively in Ref. [26] on the example of the model (1.1)–(1.3), for the pair correlation function in three dimensions, and in Ref. [24] on the example of the general \mathcal{A} model, for the correlation functions of arbitrary order in d dimensions.

The plan of the paper and the main results are the following. In Sec. II we give the field theoretic formulation of the model, its diagrammatic techniques, and derive exact equations for the response function and pair correlation function: the so-called Dyson-Wyld equations.

In Sec. III we study the inertial-range behavior of the pair correlation function in the presence of the large-scale anisotropic forcing. This issue for the model (1.1)–(1.3) was already discussed in Ref. [26], where the numerical solutions for the scaling exponents were presented in three dimensions for the isotropic sector and low-order anisotropic sectors.

In this paper, starting from the Dyson-Wyld equations, we give the general recipe of deriving nonperturbative exact equations and obtain explicitly transcendental equations for the scaling exponents, related to different irreducible representations, in d dimensions. This allows us to give general description of the behavior of the full set of solutions in isotropic and anisotropic sectors, and to derive analytical results in all sectors to order $O(\varepsilon)$ in d dimensions. Some details are given in Appendix A. These results are illustrated by a few nonperturbative solutions obtained numerically in two and three dimensions for the isotropic and low-order anisotropic sectors; later, in Sec. VID and VIE they are confirmed using the RG and OPE techniques and extended to higher-order structure functions.

In Sec. IV, we perform the UV renormalization of the model and derive the corresponding RG equations with exactly known RG functions (β function and anomalous dimensions of the basis fields and parameters). For $d^2 > 3$, these equations possess an IR stable fixed point, which establishes the existence of IR scaling with exactly known scaling dimensions of the basis fields and parameters of the model.

In Sec. V, we discuss the operator product expansion and its relationship to the issue of anomalous scaling. We show that nontrivial exponents describing the inertial-range behavior of the $2n$ th-order even structure function are related to critical dimensions of scalar composite operators built of $2n$ derivatives of the advected field. Explicit calculation of the dimensions of relevant operators is given in Sec. VI. The case $n=1$ can be treated exactly using certain functional Schwinger equation (in the case at hand, it has the meaning of the energy balance equation). Like in the scalar case [3], the function $S_2 \propto r^{2-\varepsilon}$ appears nonanomalous with the simple dimensional exponent; Sec. VIA. The case $n=2$ is studied

in detail; Sec. VIB. The critical dimensions of the relevant family of operators are calculated to order $O(\varepsilon)$ in d dimensions; they include a negative dimension, and the function S_4 shows anomalous scaling. The families of the anomalous exponents related to the higher-order functions S_{2n} are calculated in Sec. VIC, in the limit of large d , for n as high as 6; some technical details are given in Appendix B.

Generalization to the case of anisotropic sectors is discussed in Sec. VID and VIE. There, the RG and OPE techniques confirm the general picture established earlier for the pair correlation function (infinite sets of exponents, hierarchy, absence of saturation) and extend it to the case of the higher-order structure functions.

The results obtained are reviewed and discussed in the Conclusion, where the lessons one can learn regarding the stirred NS equation and possible generalization to this nonlinear problem are also briefly outlined.

II. FIELD THEORETIC FORMULATION AND THE DYSON-WYLD EQUATIONS

The stochastic problem (1.1)–(1.3) is equivalent to the field theoretic model of the extended set of three fields $\Phi \equiv \{\theta', \theta, \mathbf{v}\}$ with action functional

$$\mathcal{S}(\Phi) = \theta' D_\theta \theta' / 2 + \theta' [-\nabla_t + \nu_0 \Delta] \theta - \mathbf{v} D_v^{-1} \mathbf{v} / 2. \quad (2.1)$$

The first three terms in Eq. (2.1) represent the Martin-Siggia-Rose-type action for the stochastic problems (1.1) and (1.2) at fixed \mathbf{v} (see, e.g., [16,17] and references therein), while the last term represents the Gaussian averaging over \mathbf{v} . Here D_θ and D_v are the correlation functions (1.2) and (1.3), respectively, θ' is an auxiliary transverse vector field, the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are implied, for example,

$$\theta' \partial_t \theta \equiv \int dt d\mathbf{x} \theta'_i(t, \mathbf{x}) \partial_t \theta_i(t, \mathbf{x}).$$

The pressure term can be omitted in the functional (2.1) owing to the transversality of the auxiliary field:

$$\int dx \theta'_i \partial_i \mathcal{P} = - \int dx \mathcal{P} \partial_i \theta'_i = 0.$$

Of course, this does not mean that the pressure contribution can simply be neglected: the field θ' acts as the transverse projector and selects the transverse part of the expression in the square brackets in Eq. (2.1).

The formulation (2.1) means that statistical averages of random quantities in stochastic problems (1.1)–(1.3) can be represented as functional averages with the weight $\exp \mathcal{S}(\Phi)$, so that the generating functionals of total $[G(A)]$ and connected $[W(A)]$ Green functions of the problem are given by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[\mathcal{S}(\Phi) + A\Phi] \quad (2.2)$$

with arbitrary sources $A \equiv A^\theta, A^\theta, A^\mathbf{v}$ in the linear form

$$A\Phi \equiv \int dx [A_i^{\theta'}(x)\theta'_i(x) + A_i^\theta(x)\theta_i(x) + A_i^v(x)v_i(x)].$$

The model (2.1) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta'(\mathbf{v}\partial)\theta \equiv \theta'_i\theta_j v_k V_{ijk}$ with vertex factor

$$V_{ijk}(\mathbf{p}) = i\delta_{ij}p_k, \quad (2.3)$$

where \mathbf{p} is the momentum flowing into the vertex via the field θ' . The bare propagators in the frequency-momentum $(\omega-\mathbf{p})$ representation have the forms

$$\begin{aligned} \langle \theta_i(\omega, \mathbf{p}) \theta'_j(-\omega, -\mathbf{p}) \rangle_0 &= \langle \theta'_i(\omega, \mathbf{p}) \theta_j(-\omega, -\mathbf{p}) \rangle_0^* \\ &= \frac{P_{ij}(\mathbf{p})}{(-i\omega + \nu_0 p^2)}, \end{aligned} \quad (2.4a)$$

$$\langle \theta_i(\omega, \mathbf{p}) \theta_j(-\omega, -\mathbf{p}) \rangle_0 = \frac{C_{ij}(\mathbf{p})}{(\omega^2 + \nu_0^2 p^4)}, \quad (2.4b)$$

$$\langle \theta'_i(\omega, \mathbf{p}) \theta'_j(-\omega, -\mathbf{p}) \rangle_0 = 0, \quad (2.4c)$$

where $C_{ij}(\mathbf{p})$ is the Fourier transform of the function $C_{ij}(\mathbf{r})$ from Eq. (1.2); the bare propagator $\langle v_i v_j \rangle_0$ is given by Eq. (1.3).

The action functional (2.1) is invariant with respect to the dilatation $\theta \rightarrow \lambda\theta$, $\theta' \rightarrow \theta'/\lambda$, $C \rightarrow \lambda^2 C$, where C is the correlation function (1.2). It then follows that any total or connected Green function with n fields θ and p fields θ' is proportional to $C^{(n-p)/2}$. Since the function C appears in the bare propagator (2.4b) only in the numerators, the difference $n-p$ is an even non-negative integer for any non-vanishing function; the Green functions with $n-p < 0$ vanish identically. On the contrary, the one-irreducible function $\langle \theta(x_1) \cdots \theta(x_n) \theta'(y_1) \cdots \theta'(y_p) \rangle_{1-\text{ir}}$ contains the factor $C^{(p-n)/2}$ and therefore vanishes for $n-p > 0$; this fact will be relevant in the analysis of the renormalizability of the model (see Sec. IV).

The pair correlation functions $\langle \Phi\Phi \rangle$ of the multicomponent field Φ satisfy standard Dyson equation, which in the component notation reduces to the system of two nontrivial equations for the exact correlation function $\mathcal{D}_{ij}(\omega, \mathbf{p}) = \langle \theta_i(\omega, \mathbf{p}) \theta_j(-\omega, -\mathbf{p}) \rangle$ and the exact response function $\mathcal{G}_{ij}(\omega, \mathbf{p}) = \langle \theta'_i(\omega, \mathbf{p}) \theta'_j(-\omega, -\mathbf{p}) \rangle$. We shall see below that the latter function does not include the correlation function (1.2), therefore it is isotropic and can be written as $\mathcal{G}_{ij}(\omega, \mathbf{p}) = P_{ij}(\mathbf{p})\mathcal{G}(\omega, p)$ with certain isotropic scalar function $\mathcal{G}(\omega, p)$. Thus the component equations, usually referred to as the Dyson-Wyld equations, in our model take on the form (cf. Refs. [9] for the scalar and [23] for the magnetic models)

$$\mathcal{G}^{-1}(\omega, p)P_{ij}(\mathbf{p}) = [-i\omega + \nu_0 p^2]P_{ij}(\mathbf{p}) - \Sigma_{ij}^{\theta'\theta}(\omega, \mathbf{p}), \quad (2.5a)$$

$$\mathcal{D}_{ij}(\omega, \mathbf{p}) = |\mathcal{G}(\omega, p)|^2 [C_{ij}(\mathbf{p}) + \Sigma_{ij}^{\theta'\theta}(\omega, \mathbf{p})], \quad (2.5b)$$

where $\Sigma^{\theta'\theta}$ and $\Sigma^{\theta\theta'}$ are self-energy operators represented by the corresponding one-irreducible diagrams; the other functions $\Sigma^{\Phi\Phi}$ vanish identically. It is also convenient to contract Eq. (2.5a) with the projector $P_{ij}(\mathbf{p})$ in order to obtain the scalar equation:

$$\mathcal{G}^{-1}(\omega, p) = -i\omega + \nu_0 p^2 - \Sigma^{\theta'\theta}(\omega, p), \quad (2.6a)$$

where we have written

$$\Sigma^{\theta'\theta}(\omega, p) \equiv \Sigma_{ij}^{\theta'\theta}(\omega, \mathbf{p})P_{ij}(\mathbf{p})/(d-1). \quad (2.6b)$$

The feature characteristic of the rapid-change models like Eq. (2.1) is that all the skeleton multiloop diagrams entering into the self-energy operators $\Sigma^{\theta'\theta}$ and $\Sigma^{\theta\theta'}$ contain effectively closed circuits of retarded propagators $\langle \theta\theta' \rangle_0$ and therefore vanish; it is crucial here that the velocity propagator (1.3) contains the δ function in time and the bare propagator (2.4c) vanishes. Therefore the self-energy operators in Eq. (2.5) are given by the one-loop approximations exactly and have the forms

$$\Sigma^{\theta'\theta} = + \text{diagram}, \quad (2.7a)$$

$$\Sigma^{\theta\theta'} = + \text{diagram}. \quad (2.7b)$$

The thick solid lines in the diagrams denote the *exact* propagators $\langle \theta\theta' \rangle$ and $\langle \theta\theta \rangle$; the ends with a slash correspond to the field θ' , and the ends without a slash correspond to θ ; the wavy lines denote the velocity propagator (1.3); the vertices correspond to the factor (2.3). The first equation does not include the correlation function (1.2), which justifies the isotropic form of the function \mathcal{G}_{ij} . The analytic expressions for the diagrams in Eq. (2.7) have the forms

$$\begin{aligned} \Sigma^{\theta'\theta}(\omega, p) &= \frac{P_{ij}(\mathbf{p})}{(d-1)} \int \mathcal{D}\omega' \int \mathcal{D}\mathbf{k} V_{ii_3 i_1}(\mathbf{p}) P_{i_3 i_4}(\mathbf{p}-\mathbf{k}) \\ &\quad \times \mathcal{G}(\omega', \mathbf{p}-\mathbf{k}) \frac{D_0 P_{i_1 i_2}(\mathbf{k})}{k^{d+\varepsilon}} V_{i_4 j i_2}(\mathbf{p}), \end{aligned} \quad (2.8a)$$

$$\begin{aligned} \Sigma^{\theta\theta'}(\omega, \mathbf{p}) &= \int \mathcal{D}\omega' \int \mathcal{D}\mathbf{k} V_{ii_3 i_1}(\mathbf{p}) \mathcal{D}_{i_3 i_4}(\omega', \mathbf{p}-\mathbf{k}) \\ &\quad \times \frac{D_0 P_{i_1 i_2}(\mathbf{k})}{k^{d+\varepsilon}} V_{j i_4 i_2}(-\mathbf{p}). \end{aligned} \quad (2.8b)$$

Here we have denoted $\mathcal{D}\omega' \equiv d\omega'/(2\pi)$, used the explicit form (1.3) of the velocity covariance and the relation $P_{i_1 i_2}(\mathbf{k})V_{i_4 j i_2}(\mathbf{p}-\mathbf{k}) = P_{i_1 i_2}(\mathbf{k})V_{i_4 j i_2}(\mathbf{p})$ for the vertex factor in Eq. (2.3). We also recall that the integrations over \mathbf{k} should be cut off from below at $k=m$.

The integrations with respect to ω' on the right-hand sides of Eqs. (2.8) give the equal-time response function $\mathcal{G}(k) = \int \mathcal{D}\omega' \mathcal{G}(\omega', k)$ and the equal-time pair correlation function $\mathcal{D}_{ij}(\mathbf{k}) = \int \mathcal{D}\omega' \mathcal{D}_{ij}(\omega', \mathbf{k})$; note that both the self-

energy operators appear independent of ω . The only contribution to \mathcal{G} comes from the bare propagator (2.4a), which in the t representation is discontinuous at coincident times. Since the correlation function (1.3), which enters into the one-loop diagram for $\Sigma^{\theta'\theta}$, is symmetric in t and t' , the response function must be defined at $t=t'$ by half the sum of the limits, which is equivalent to the convention $\mathcal{G}(k) = \int \mathcal{D}\omega' (-i\omega' + \nu_0 k^2)^{-1} = 1/2$. This allows one to write the equation (2.6a) in the form

$$\mathcal{G}^{-1}(\omega, p) = [-i\omega + \nu_{eff}(p)p^2], \quad (2.9)$$

where the p -dependent effective ‘‘eddy diffusivity’’ is given by

$$2p^2 \nu_{eff}(p) \equiv 2\nu_0 p^2 + D_0 \int \frac{D\mathbf{k}}{k^{d+\varepsilon}} \left[1 - \frac{(\mathbf{p}\mathbf{k})^2}{p^2 k^2} \right] \times \left[p^2 - \frac{p^2 k^2 - (\mathbf{p}\mathbf{k})^2}{(d-1)|\mathbf{p}-\mathbf{k}|^2} \right]. \quad (2.10)$$

It follows from Eq. (2.10) that the eddy diffusivity can be written as the sum of two parts: $\nu_{eff}(p) = \nu_{loc} + \nu_{non}(p)$, where the local part is p independent and coincides with the expression for the effective diffusivity known in the scalar and magnetic cases, while the nonlocal contribution has a finite limit at $m=0$ but retains a nontrivial dependence on the momentum:

$$\nu_{loc} = \nu_0 + \frac{D_0}{2} \int \frac{D\mathbf{k}}{k^{d+\varepsilon}} \left[1 - \frac{(\mathbf{p}\mathbf{k})^2}{p^2 k^2} \right] = \nu_0 + D_0 C_d m^{-\varepsilon} \frac{(d-1)}{2d\varepsilon}, \quad (2.11a)$$

$$\nu_{non}(p) = \frac{D_0}{2} \int \frac{D\mathbf{k}}{k^{d+\varepsilon}} \left[1 - \frac{(\mathbf{p}\mathbf{k})^2}{p^2 k^2} \right] \frac{(\mathbf{p}\mathbf{k})^2 - p^2 k^2}{(d-1)p^2 |\mathbf{p}-\mathbf{k}|^2}. \quad (2.11b)$$

Here and below $C_d \equiv S_d / (2\pi)^d$ and $S_d \equiv 2\pi^{d/2} / \Gamma(d/2)$ is the surface area of the unit sphere in d -dimensional space and $\Gamma(z)$ is the Euler Gamma function. The parameter m in ν_{loc} has arisen from the lower limit in the integral over \mathbf{k} . For $m=0$, Eq. (2.11b) gives

$$\nu_{non}(p) = -D_0 p^{-\varepsilon} J / (4\pi)^{d/2}, \quad (2.12)$$

where

$$J = \frac{(d+1)\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(1 - \frac{\varepsilon}{2}\right)\Gamma\left(1 + \frac{d}{2}\right)}{8\Gamma\left(1 + \frac{d+\varepsilon}{2}\right)\Gamma\left(2 + \frac{d-\varepsilon}{2}\right)}, \quad (2.13)$$

while for $p=0$ one obtains

$$\nu_{non} = -C_d D_0 m^{-\varepsilon} \frac{(d+1)}{2d(d+2)\varepsilon},$$

$$\nu_{eff} = \nu_0 + C_d D_0 m^{-\varepsilon} \frac{(d^2-3)}{2d(d+2)\varepsilon}. \quad (2.14)$$

Equations (2.9)–(2.14) give the explicit exact expression for the response function in our model; it will be used in Sec. IV for the exact calculation of the RG functions. The integration of Eq. (2.5b) over the frequency ω gives a closed equation for the equal-time correlation function; it is important here that the ω dependence of the right-hand side is contained only in the prefactor $|\mathcal{G}(\omega, p)|^2$:

$$2\nu_{eff}(p)p^2 \mathcal{D}_{ij}(\mathbf{p}) = C_{ij}(\mathbf{p}) + \Sigma_{ij}^{\theta'\theta'}(\mathbf{p}). \quad (2.15)$$

Using Eqs. (2.8b), (2.10), and (2.11), the equation for \mathcal{D}_{ij} can be rewritten in the form

$$2\nu_0 p^2 \mathcal{D}_{ij}(\mathbf{p}) = C_{ij}(\mathbf{p}) + p^2 D_0 \int \frac{D\mathbf{k}}{k^{d+\varepsilon}} \left[1 - \frac{(\mathbf{p}\mathbf{k})^2}{p^2 k^2} \right] \times \{ P_{ii_1}(\mathbf{p}) \mathcal{D}_{i_1 i_2}(\mathbf{p}-\mathbf{k}) P_{i_2 j}(\mathbf{p}) - \mathcal{D}_{ij}(\mathbf{p}) \} - 2\nu_{non}(p)p^2 \mathcal{D}_{ij}(\mathbf{p}). \quad (2.16)$$

The subtracted term in the curly brackets is the contribution of ν_{loc} in Eq. (2.15) written in its integral form (2.11a). For $0 < \varepsilon < 2$, the IR cutoff in Eq. (2.16) can be removed. Indeed, owing to the subtraction, the integral over \mathbf{k} is finite for $m \rightarrow 0$: the possible IR divergence at $\mathbf{k}=0$ is suppressed by the expression in the curly brackets. In what follows we set $m=0$ in Eq. (2.16).

It is instructive to compare expression (2.14) for the effective diffusivity in our model with its analogs for the scalar [4] and magnetic [23] cases. For those, the nonlocal part of the effective diffusivity vanishes identically, while its local part coincides with Eq. (2.11a). Therefore the ratio $\nu_{loc}/\nu_{eff} = (d-1)(d+2)/(d^2-3)$ (we have put $p=0$ and neglected the bare diffusivity ν_0) can be considered as a measure of the nonlocality contribution into the turbulent diffusion. It tends to unity as $d \rightarrow \infty$, increases monotonically as d decreases and diverges at $d^2=3$. This means that the nonlocality contribution is negligible at large d (see also Ref. [24] for the general vector model), becomes comparable with the local contribution as d is reduced, and dominates the diffusion in low dimensions (in particular, $\nu_{loc}/\nu_{eff} = 5/3$ and 4 for $d=3$ and 2, respectively).

We also notice that, according to Eq. (2.14), the effective diffusivity $\nu_{eff}(p)$ becomes *negative* for small $p \ll m$ and $d^2 < 3$. Therefore the response function in the time-momentum representation,

$$\mathcal{G}_{ij}(t, \mathbf{p}) = \Theta(t) P_{ij}(\mathbf{p}) \exp\{-\nu_{eff}(p)p^2 t\},$$

where $\Theta(t)$ is the step function, grows with time for small \mathbf{p} , thus signalling that the steady-state solution cannot be stable for $d^2 < 3$: any small perturbation would lead to the exponential in-time growth of the mean field $\langle \theta \rangle$. Indeed, one can easily see that Eq. (2.15) for the correlation function has no solutions at small \mathbf{p} and $d^2 < 3$: its left-hand side is negative, while the right-hand side is strictly positive for all d . We shall see below in Sec. III A that the inertial-range solutions of Eq. (2.15) also become singular and disappear in the limit $d^2 \rightarrow 3$.

Although the instability occurs for unphysical value of d , it deserves some attention as a result of the competition between the local and nonlocal contributions: from Eqs. (2.11) it follows that ν_{loc} is strictly *positive* for all d , while ν_{non} is strictly *negative*. In a few respects, such an instability differs from that established in Ref. [27] for the magnetic (local) case, in three dimensions, where $\nu_{non}=0$, the effective diffusivity coincides with its local part (2.11a) and is always positive. Thus the response function, and hence the mean $\langle \theta \rangle$, show no hint of misbehavior at the threshold of the instability; the latter reveals itself on the level of the pair correlation function and can be related to the complexification of the inertial-range exponents; see Refs. [27–30].

III. INERTIAL-RANGE BEHAVIOR AND SCALING EXPONENTS FOR THE PAIR CORRELATION FUNCTION IN THE PRESENCE OF LARGE-SCALE ANISOTROPY

It is well known [4–6] that nontrivial inertial-range exponents are determined by zero modes, i.e., the solutions of Eq. (2.16) neglecting both the forcing [$C(\mathbf{r})=0$] and the dissipation ($\nu_0=0$). Whatever be the forcing, equations for zero modes are linear and $SO(d)$ covariant, and their solution can be sought in the form of decomposition in irreducible representations of the rotation group. Equation (2.16) then falls into independent equations for the coefficient functions. In three dimensions, such decompositions were used in Refs. [26,30].

Below we use more elementary derivation, which allows one to obtain explicitly transcendental equations for the scaling exponents, related to different irreducible representations, in d dimensions. We restrict ourselves to the case of *uniaxial* anisotropy, specified by a unit vector \mathbf{n} , which is sufficient to find *all* independent exponents, and use explicit expressions for the basis functions in the *momentum representation*. Then the transversality condition can be easily imposed from the very beginning, and there is no need to check it *a posteriori*.

In contrast with the real-space Legendre decomposition, employed in Refs. [23,29], our representation is consistent with the rotational symmetry: it can be considered as the projection of the complete $SO(d)$ decomposition onto the subspace of the functions with uniaxial symmetry. Therefore, no additional assumptions, like the hierarchy of exponents, are needed to disentangle the equations for different anisotropic sectors.

We start with the isotropic case and then discuss the general situation.

A. Solution in the isotropic shell

In the isotropic case, the inertial-range solution is sought in the form

$$\mathcal{D}_{ij}(\mathbf{p}) = A P_{ij}(\mathbf{p}) p^{-d-\gamma}. \quad (3.1)$$

The zero-mode analog of Eq. (2.16) takes on the form

$$D_0 \int \frac{\mathcal{D}\mathbf{k}}{k^{d+\varepsilon}} \left[1 - \frac{(\mathbf{p}\mathbf{k})^2}{p^2 k^2} \right] \left\{ \left[\frac{1}{|\mathbf{p}-\mathbf{k}|^{d+\gamma}} - \frac{1}{p^{d+\gamma}} \right] - \frac{k^2 [1 - (\mathbf{p}\mathbf{k})^2 / p^2 k^2]}{(d-1)|\mathbf{p}-\mathbf{k}|^{d+\gamma+2}} \right\} = 2\nu_{non}(p) p^{-d-\gamma}. \quad (3.2)$$

In the following, we shall need the standard reference integrals

$$\int \mathcal{D}\mathbf{k} \frac{[1 - (\mathbf{p}\mathbf{k})^2 / p^2 k^2]^n}{k^{d+\alpha} |\mathbf{p}-\mathbf{k}|^{d+\beta}} \equiv I_n(\alpha, \beta) \frac{p^{-d-\alpha-\beta}}{(4\pi)^{d/2}}, \quad (3.3a)$$

where

$$I_n(\alpha, \beta) = \frac{\Gamma\left(n + \frac{d-1}{2}\right) \Gamma\left(-\frac{\alpha}{2}\right) \Gamma\left(n - \frac{\beta}{2}\right) \Gamma\left(\frac{\alpha+\beta+d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right) \Gamma\left(n + \frac{\alpha+d}{2}\right) \Gamma\left(\frac{\beta+d}{2}\right) \Gamma\left(n - \frac{\alpha+\beta}{2}\right)}. \quad (3.3b)$$

The integral (3.3a) is finite in the region of parameters specified by the inequalities $\alpha < 0$ (convergence at $\mathbf{k} \rightarrow 0$), $\beta < 2n$ (convergence at $\mathbf{k} - \mathbf{p} \rightarrow 0$, improved by the factor $[1 - (\mathbf{p}\mathbf{k})^2 / p^2 k^2]^n \propto |\mathbf{k} - \mathbf{p}|^{2n}$) and $\alpha + \beta > -d$ (convergence at $\mathbf{k} \rightarrow \infty$). However, expression (3.3b) is meaningful in a wider range of parameters and, in the spirit of analytical and dimensional regularizations [37,38], it can be considered as the analytical continuation of the integral (3.3a) from the region in which it converges. The precise meaning of such a continuation is that Eq. (3.3) gives the correct value of the integral with proper subtractions that ensure its convergence. For example, if the factor $1/|\mathbf{p}-\mathbf{k}|^{d+\beta}$ is replaced with the difference $1/|\mathbf{p}-\mathbf{k}|^{d+\beta} - 1/p^{d+\beta}$ (that is, the zeroth term of the Taylor expansion in \mathbf{k} is subtracted), the integral becomes convergent for $\alpha < 2$ and expression Eq. (3.3) gives the correct answer for this “improved” integral.

One can easily see that Eq. (3.2) involves such a subtraction, which improves its convergence at small \mathbf{k} . Therefore, one can use the formal expression (3.3) in Eq. (3.2) and simultaneously omit the subtracted term. Then Eq. (3.2) takes the form

$$I_1(\varepsilon, \gamma) - I_2(\varepsilon - 2, \gamma + 2) / (d-1) = -2J, \quad (3.4)$$

with J from Eq. (2.13). We shall see below that the *leading admissible* solution of this equation is $\gamma = 2 - \varepsilon$, so that the integrals entering into Eq. (3.2) are convergent for all $0 < \varepsilon < 2$ and the above procedure is internally consistent. In Eq. (3.4), we omit the overall nonvanishing factor

$$\frac{\Gamma\left(1 - \frac{\varepsilon}{2}\right) \Gamma\left(1 - \frac{\gamma}{2}\right) \Gamma\left(\frac{\gamma+d+\varepsilon}{2}\right)}{\varepsilon(d+1) \Gamma\left(1 + \frac{d+\varepsilon}{2}\right) \Gamma\left(1 + \frac{d+\gamma}{2}\right) \Gamma\left(2 - \frac{\gamma+\varepsilon}{2}\right)}$$

and obtain the desired equation for the zero-mode exponents in the isotropic case:

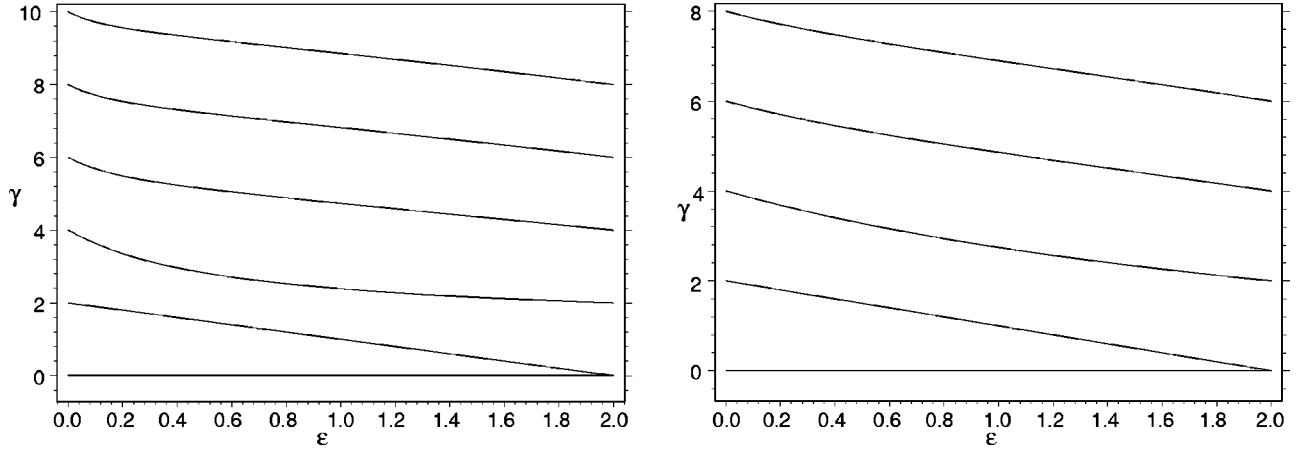


FIG. 1. Leading scaling exponents for the isotropic sector $l=0$ in $d=2$ (left) and $d=3$ (right).

$$\begin{aligned}
 & (d-1)(d+\gamma)(2-\gamma-\varepsilon)/(d+1)+\varepsilon \\
 & = 2 \frac{\Gamma\left(1+\frac{d}{2}\right)\Gamma\left(1+\frac{\varepsilon}{2}\right)\Gamma\left(1+\frac{d+\gamma}{2}\right)\Gamma\left(2-\frac{\varepsilon+\gamma}{2}\right)}{\Gamma\left(2+\frac{d-\varepsilon}{2}\right)\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma\left(\frac{d+\varepsilon+\gamma}{2}\right)}. \quad (3.5)
 \end{aligned}$$

The transcendental equation (3.5) has infinitely many solutions. This means that the inertial-range behavior of the correlation function is given by an infinite sum of powerlike contributions of the form (3.1); the leading term is given by the minimal exponent γ .

Some solutions can be ruled out as not admissible [4,5]; admissible solutions are non-negative for $\varepsilon=0$ (see, e.g., [23,30]). The remaining solutions, all having the forms $\gamma = -d - 2k + O(\varepsilon)$ with non-negative integer k , are also meaningful and describe the behavior of the correlation function at large scales $r \gg L$ [4]; we shall not discuss them in the following.

It then follows from Eq. (3.5) that the leading admissible inertial-range solution is $\gamma = 2 - \varepsilon$ (no corrections of order ε^2 and higher). In the coordinate representation, this corresponds to $S_2 \propto r^{2-\varepsilon}$, that is, the second-order structure function is nonanomalous like in the scalar model [3].

For small ε , all the subleading exponents can be written in the form

$$\begin{aligned}
 \gamma_k &= 2k - \varepsilon \frac{(d-1)(d+2)}{(d^2-3)} + \varepsilon^2 \frac{(d+1)(d+2)}{2(d^2-3)^2} \\
 & \times \left\{ (d-1)\mathcal{K} - \frac{(d+1)}{(k-1)(d+2k)} \right\} + O(\varepsilon^3) \quad (3.6)
 \end{aligned}$$

with $k=2,3,4$ and so on. The functions $\psi(z) = d \ln \Gamma(z)/dz$ can be eliminated from the coefficient

$$\mathcal{K} = \psi(k+d/2) - \psi(2+d/2) + \psi(k) - \psi(1)$$

using the relation $\psi(z+1) = \psi(z) + 1/z$. In order to prevent the appearance of ambiguities in the zeroth order in ε , it is

convenient to rewrite the right-hand side of Eq. (3.5) such that the Gamma functions have no poles at $\gamma_k = 2k$. This is easily done using the relation $\Gamma(1+z)\Gamma(1-z) = \pi z / \sin(\pi z)$ and is also useful for the numerical solution.

Nonperturbative solutions of Eq. (3.5) can only be obtained numerically. They are illustrated by Fig. 1 for $d=2$ (left) and $d=3$ (right); the latter is in agreement with Fig. 2 from Ref. [26]. One can see that solutions (3.6) exist for all $0 \leq \varepsilon \leq 2$, decrease monotonically as ε grows and turn to $\gamma_k = 2k - 2$ at $\varepsilon = 2$. The exponent $\gamma = 0$ corresponds to the solution $\delta_{ij} \delta(\mathbf{p})$ that exists for all d (see below).

It is easily seen that the integrals entering into Eq. (3.2) are divergent on these solutions at $\mathbf{k} - \mathbf{p} \rightarrow 0$. One can say that the knowledge of the exponents is insufficient to discuss the convergence: it is necessary to know the behavior of the entire solution in the region of small momenta, $|\mathbf{k} - \mathbf{p}| \ll m$, where it no longer reduces to a sum of power terms with the exponents (3.6). However, it is intuitively clear that the form of the solution at such small momenta is irrelevant for the calculation of the inertial-range exponents. Indeed, we made no assumptions about the form of the solution in that range, sought them in purely powerlike form and managed to derive closed equations for the exponents using the prescriptions of the analytical regularization. A simple justification of this procedure follows.

In coordinate representation, the solution is sought in the form $\mathcal{D}(\mathbf{r}) \propto r^\gamma \mathcal{C}(m\mathbf{r})$ (for simplicity, here and below we omit its vector indices). The convergence problems arise if $\gamma > 0$, which is implied in what follows. It is natural to assume that the scaling function $\mathcal{C}(m\mathbf{r})$ is such that $\mathcal{D}(\mathbf{r})$ vanishes at $\mathbf{r} = 0$ along with all its derivatives up to the n th order, where n is the maximal integer satisfying the inequality $\gamma > n$. This gives the set of integral relations

$$\int d\mathbf{q} q_{i_1} \dots q_{i_k} \mathcal{D}(\mathbf{q}) = 0, \quad k=0,1,\dots,n \quad (3.7)$$

for the correlation function $\mathcal{D}(\mathbf{q}) \propto q^{-d-\gamma} \tilde{\mathcal{C}}(\mathbf{q}/m)$ in momentum representation. The integral entering into Eq. (3.2) can symbolically be written as

$$\int d\mathbf{q} F(\mathbf{q}, \mathbf{p}) \mathcal{D}(\mathbf{q}), \quad (3.8)$$

where we have introduced the new variable $\mathbf{q} = \mathbf{p} - \mathbf{k}$. The form of the kernel $F(\mathbf{q}, \mathbf{p})$ is clear from the comparison with Eq. (3.2); the divergence can arise from the region $\mathbf{q} \rightarrow 0$, where the solution behaves as $\mathcal{D}(\mathbf{q}) \propto q^{-d-\gamma}$. Owing to relations (3.7), the value of integral (3.8) does not change if one subtracts from $F(\mathbf{q}, \mathbf{p})$ the first terms of its Taylor expansion up to the n th order:

$$\int d\mathbf{q} \left\{ F(\mathbf{q}, \mathbf{p}) - F(0, \mathbf{p}) - q_i \frac{\partial F(0, \mathbf{p})}{\partial q_i} - \dots - \frac{1}{n!} q_{i_1} \dots q_{i_n} \frac{\partial^n F(0, \mathbf{p})}{\partial q_{i_1} \dots \partial q_{i_n}} \right\} \mathcal{D}(\mathbf{q}). \quad (3.9)$$

Now one can set $m=0$ in the function $\mathcal{D}(\mathbf{q})$, that is, replace the exact solution with its inertial-range asymptotic expression $\mathcal{D}(\mathbf{q}) \propto q^{-d-\gamma}$: the possible divergence at $\mathbf{q}=0$ is suppressed by the expression in the curly brackets, which behaves as q^{n+2} for $\mathbf{q} \rightarrow 0$. Therefore, one can use the formal rules of analytical regularization and simultaneously omit the subtracted terms [38]: this gives the correct answer for the convergent integral with proper subtractions in Eq. (3.9).

We thus conclude that the exponents (3.6) may appear in the full solution as correction terms; in the corresponding integrals exact solution can be replaced with its powerlike asymptote and the resulting integrals are properly given by the rules of analytical regularization.

Our conclusions are in agreement with those drawn in Ref. [26] for the equation in coordinate space, although the analysis in momentum space appears rather different. In particular, the momentum-space analysis reveals the close resemblance between the scalar and vector models: for the former, the correlation function in momentum representation also satisfies an *integral* equation and the above discussion is needed to fix the convergence problem. It is also worth noting that the procedure employed in Ref. [26] for the calculation of divergent integrals involves analytical continuation from the region of convergence, and is therefore close to the concept of analytical regularization.

Furthermore, the ‘‘realizability’’ of solutions (3.6) is guaranteed by the fact that in the RG approach they are identified with the critical dimensions of composite operators entering into the corresponding operator product expansions; see Sec. V.

Besides powerlike solutions, Eq. (3.2) also possesses the solution of the form $\delta_{ij} \delta(\mathbf{p})$. To demonstrate this, we use the well-known representation of the d -dimensional δ function

$$\begin{aligned} \delta(\mathbf{p}) &= \lim_{\sigma \rightarrow 0} \int \mathcal{D}\mathbf{x} (\Lambda x)^{-\sigma} \exp[i(\mathbf{p}\mathbf{x})] \\ &= S_d^{-1} p^{-d} \lim_{\sigma \rightarrow 0} [\sigma (p/\Lambda)^\sigma] \end{aligned} \quad (3.10)$$

and substitute it into Eq. (3.2). For small σ , the integral on its left-hand side is finite. Therefore it vanishes as $\sigma \rightarrow 0$ in

Eq. (3.10), and the equation is satisfied. Another such solution, $[\delta_{ij} - dn_i n_j] \delta(\mathbf{p})$, belongs to the first anisotropic sector. The both solutions are automatically transverse owing to the relation $p_i \delta(\mathbf{p}) = 0$. In coordinate representation, they correspond to constant terms, so that the exponent $\gamma=0$ can formally be assigned to them. They are indeed present in the pair correlation function, but disappear from the structure function (1.6) owing to the relation $S_2=0$ for $\mathbf{r}=0$.

We have already established in the end of Sec. II that the model (1.1)–(1.3) becomes unstable at $d = \sqrt{3}$ and no steady-state solution for the correlation function exists in the range of small momenta, $p \ll m$. It is easily seen that expressions (3.6) for the inertial-range exponents diverge in this limit. The same singularity occurs in the first-order expressions for another exponents in our model; see, e.g., Eqs. (3.21)–(3.25), (6.25), and (6.33) in subsequent sections. Moreover, the RG analysis shows that the actual expansion parameter is $g_* \propto \varepsilon/(d^2-3)$ rather than ε itself; see Eq. (4.9) in Sec. IV. Therefore, the higher-order terms of the ε expansions become more and more singular as d approaches $\sqrt{3}$ from above and any finite-order approximation cannot be trusted.

Numerical solution shows that for d close to the threshold, the behavior of exponents γ_k consists of two pronounced stages. At the beginning, the exponents decrease rapidly as ε increases in agreement with the first-order expression in Eq. (3.6). Then, for a very small value of ε (which tends to zero as $d \rightarrow \sqrt{3}$) the instantaneous stabilization takes place at an almost constant value $\gamma_k \approx 2k-2$, and for $\varepsilon=2$ one has $\gamma_k = 2k-2$ exactly.

For $d = \sqrt{3}$ the solutions do not exist. This fact can be naturally explained, and extended to the other exponents, on the basis of the ε expansion. The series in ε can be rewritten as series in $g_* \propto \varepsilon/(d^2-3)$, with coefficients regular at $d^2=3$. Then they can be inverted into expansions of g_* in γ [more precisely, in $\gamma-2k=O(\varepsilon)$], where the coefficients are also regular. In other words, the inverted ε series can be written as $\varepsilon = (d^2-3)f(\gamma)$, where f is a function regular as $d \rightarrow \sqrt{3}$. For $d = \sqrt{3}$ this gives $\varepsilon=0$, that is, no solution exists for γ if $\varepsilon > 0$.

B. Scaling behavior in anisotropic sectors

Now let us turn to the case of uniaxial anisotropy, specified by a unit vector \mathbf{n} in the correlation function (1.2), and introduce the following irreducible l th rank tensors

$$\mathcal{N}_{i_1 \dots i_l}^{(l)} \equiv \mathcal{P}_{\text{irr}}[n_{i_1} \dots n_{i_l}]. \quad (3.11)$$

Here and below, \mathcal{P}_{irr} denotes the irreducible part, obtained by subtracting the appropriate expression involving the Kronecker δ symbols, such that the resulting tensor is traceless with respect to any pair of indices. In particular, $\mathcal{N}_i^{(1)} = n_i$, $\mathcal{N}_{ij}^{(2)} = n_i n_j - \delta_{ij}/d$, $\mathcal{N}_{ijk}^{(3)} = n_i n_j n_k - (\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk} n_i)/(d+2)$ and so on.

It is easy to see that structures (3.11) are orthogonal on the sphere:

$$\int d\mathbf{n} \mathcal{N}^{(l)} \mathcal{N}^{(s)} = 0 \quad \text{for } l \neq s, \quad (3.12)$$

where $d\mathbf{n}$ is the area element of the unit sphere in the d -dimensional space.

The contraction of the tensor (3.11) with a fixed vector \mathbf{p} gives

$$\mathcal{N}_{i_1 \dots i_l}^{(l)} p_{i_1} \dots p_{i_l} = p^l P_l(z), \quad z \equiv (\mathbf{np})/p, \quad (3.13)$$

where $P_l(z) = z^l + O(z^{l-2})$ are the Gegenbauer polynomials, which reduce to the Legendre polynomials and trigonometrical functions in three and two dimensions, respectively [39].

The relation (3.12) implies that these polynomials are orthogonal on the sphere even if their arguments are different:

$$\int d\mathbf{n} P_l(z) P_s(z') = 0 \quad \text{for } l \neq s, \quad (3.14)$$

where $z \equiv (\mathbf{np})/p$ and $z' \equiv (\mathbf{np}')/p'$ with any fixed vectors \mathbf{p} and \mathbf{p}' .

In terms of the structures (3.11) and transverse projectors P_{ij} the desired decomposition of the pair correlation function can be written as follows:

$$\mathcal{D}_{ij}(\mathbf{p}) = \sum_{l=0}^{\infty} \mathcal{D}_{ij}^{(l)}(\mathbf{p}), \quad (3.15)$$

where the summation runs over all even values of l and the coefficient functions have the forms

$$\begin{aligned} \mathcal{D}_{ij}^{(l)}(\mathbf{p}) = & P_{ii}(\mathbf{p}) [A^{(l)}(p) \delta_{i_1 i_2} (\mathcal{N}_{i_3 i_4 \dots i_{l+2}}^{(l)} p_{i_3} p_{i_4} \dots p_{i_{l+2}}) \\ & + (B^{(l)}(p) p^2 \mathcal{N}_{i_1 i_2 i_3 i_4 \dots i_l}^{(l)} p_{i_3} p_{i_4} \dots p_{i_l})] P_{i_2 j}(\mathbf{p}). \end{aligned} \quad (3.16)$$

Whatever be the coefficient functions $A^{(l)}$ and $B^{(l)}$, dependent only on $p = |\mathbf{p}|$, the expression (3.16) is symmetric in the tensor indices i, j and orthogonal to the vector \mathbf{p} : $p_i \mathcal{D}_{ij}^{(l)}(\mathbf{p}) = 0$.

The first tensor structure in Eq. (3.16) can be expressed in terms of the Gegenbauer polynomials using the relation (3.13), while the second structure can be expressed in terms of the polynomials $P_l(z)$ and their derivatives using the relation

$$l(l-1) \mathcal{N}_{i_1 i_2 i_3 i_4 \dots i_l}^{(l)} p_{i_3} p_{i_4} \dots p_{i_l} = \frac{\partial^2}{\partial p_{i_1} \partial p_{i_2}} [p^l P_l(z)],$$

which is obvious from the definition (3.11) and the relation (3.13).

Substituting the series (3.15) into the zero-mode analog of Eq. (2.16) then gives the equation for the coefficient functions $A^{(l)}$ and $B^{(l)}$, which can be symbolically written as $\Lambda_{ij}(\mathbf{p}) = 0$. Its left-hand side can be decomposed using the same tensor structures (3.11), so that the equation can be reduced to an infinite family of the scalar equations

$$\int d\mathbf{n} \Lambda_{ii}(\mathbf{p}) (\mathbf{pn})^l = 0, \quad \int d\mathbf{n} \Lambda_{ij}(\mathbf{p}) \mathbf{n}_i \mathbf{n}_j (\mathbf{pn})^{l-2} = 0, \quad (3.17)$$

with the integration over the unit sphere; cf. Eq. (3.12). Owing to the generalized orthogonality relation (3.14), the l th pair of equations (3.17) involves only the coefficient functions $A^{(l)}$ and $B^{(l)}$ with the same index l , so that the equations for different values of l have decoupled.

In the inertial range, the solution in the l th sector is sought in the form $A^{(l)} = a_l p^{-d-l-\gamma_l}$, $B^{(l)} = b_l p^{-d-l-\gamma_l}$, which corresponds to $\mathcal{D}^{(l)}(\mathbf{r}) \propto r^{\gamma_l}$ in coordinate representation; for the isotropic shell, $l=0$, this reduces to the single term (3.1). For $l \geq 2$, one obtains a pair of linear equations for each pair of coefficients a_l, b_l :

$$C_{11}^{(l)} a_l + C_{12}^{(l)} b_l = 0, \quad C_{21}^{(l)} a_l + C_{22}^{(l)} b_l = 0, \quad (3.18)$$

where the coefficients $C_{\alpha\beta}^{(l)}$ depend on ε, d and the unknown exponent γ_l . They all can be expressed in terms of the basic integrals (3.3). In particular, for $l=2$ one obtains

$$C_{11}^{(2)} = (d-1)^2 \tilde{I}_1 - (d^2-1) I_2 + d I_3,$$

$$C_{12}^{(2)} = C_{21}^{(2)} = -(d-1) \tilde{I}_1 + (d+1) I_2 - d I_3,$$

$$C_{22}^{(2)} = (d-1)(d^2-2) \tilde{I}_1 / 2 - (d^2-1) I_2 + d I_3, \quad (3.19)$$

where $\tilde{I}_1 \equiv I_1 + 2J$ with J from Eq. (2.13) and $I_n \equiv I_n(\gamma_2 - 2, \varepsilon + 2)$. The l th pair of equations involves integrals $I_n \equiv I_n(\gamma_l - 2, \varepsilon + 2)$ with n as high as $l/2 + 2$. The coefficients $C_{\alpha\beta}^{(l)}$ for higher values of l up to $l=12$ are given in Appendix A.

The desired equation for γ_l is obtained as the requirement that the linear homogeneous system (3.18) have nontrivial solutions:

$$\det |C_{\alpha\beta}^{(l)}| = 0. \quad (3.20)$$

For any given l , the determinant can easily be written down in terms of standard integrals, provided the coefficients are known [see Eq. (3.19) for $l=2$ and Appendix A for $l \leq 12$]; then using the explicit expressions (3.3) for I_n one obtains transcendental equations similar to Eq. (3.5) but much more cumbersome. We shall not write them down explicitly for the sake of brevity and turn to the corresponding solutions.

It is also worth noting that for $l \geq 4$ the formal divergence of the integrals in Eq. (3.2) occurs already for the leading exponents and the discussion, similar to that given in Sec. III A, is needed to justify the use of analytic regularization.

For $d=2$, the two structures in Eq. (3.16) coincide and the determinant $\det |C_{\alpha\beta}^{(l)}|$ vanishes identically. We shall return to this case in the end of the section, and from now on we assume $d \neq 2$.

For any given $l \geq 2$ and small ε , all possible solutions for the exponents γ_l can be written in the form $\gamma_l = (l-2 + 2k) + O(\varepsilon)$ with $k=0, 1, 2, \dots$. The leading exponent corresponds to $k=0$; it is unique for any $l \geq 2$ and has the form

$$\gamma_l = (l-2) \left\{ 1 + \varepsilon \times \frac{(d+2)(l-3)(d^2-4d+2ld+l^2-5l+4)}{(d^2-3)(d+2l-6)(d+2l-4)(d+2l-2)} + O(\varepsilon^2) \right\}. \quad (3.21)$$

For all $k \geq 1$, there are exactly two solutions. For $k=1$, they can be written as

$$\gamma_l = l - \varepsilon \frac{(d+2)x_{\pm}}{(d^2-3)(d+2l-2)} + O(\varepsilon^2), \quad (3.22)$$

where the slopes x_{\pm} satisfy the quadratic equation

$$\begin{aligned} x^2(d+2l-4)(d+2l) - [d^4 + (4l-5)d^3 + 4(l^2-4l+2)d^2 \\ + (-4l^3-2l^2+14l-4)d - 2l(l-1)(l-2)(3l+1)]x \\ - l(l-1)(d+l-1)[d^3 + (3l-5)d^2 \\ + (2l^2-11l+8)d + 2(-l^3+l^2+3l-2)] = 0. \end{aligned}$$

In particular, for $l=2$ this gives

$$\begin{aligned} x_{\pm} = \{d^3 + 3d^2 - 8d - 16 \\ \pm \sqrt{(d+4)(d^5 + 2d^4 - 7d^3 - 4d^2 + 8d + 16)}\} / 2(d+4). \end{aligned} \quad (3.23)$$

For $k=2$, the solutions have the form

$$\begin{aligned} \gamma_l = (l+2) + \varepsilon \frac{(d+2)[-d^4 + (2-6l)d^3 + (3+9l-11l^2)d^2 + (-6+6l+16l^2-6l^3)d + l(l^2-1)(l+10)]}{(d^2-3)(d+2l-2)(d+2l)(d+2l+2)} + O(\varepsilon^2), \\ \gamma_l = (l+2) - \varepsilon \frac{(d-1)(d+2)}{(d^2-3)} + O(\varepsilon^2). \end{aligned} \quad (3.24)$$

The situation simplifies for all $k \geq 3$: then the solutions in order $O(\varepsilon)$ are degenerate and have the form

$$\gamma_l = (l-2+2k) - \varepsilon \frac{(d-1)(d+2)}{(d^2-3)} + O(\varepsilon^2), \quad (3.25)$$

that is, the slope is the same as for the second solution in Eq. (3.24) and the standard slope (3.6) for the isotropic sector. However, the degeneracy is removed by the $O(\varepsilon^2)$ terms, and the two solutions (3.25) do not coincide identically.

At the opposite edge, $\varepsilon=2$, all the solutions can also be found analytically for any given $l \geq 2$. They take on the values $l-2$ (single), l , $l+2$, $l+4$, and so on (twofold degeneracy for $d \neq 2$ and single otherwise). In addition to these “standard” values, for all $l \geq 2$ there are exactly two d -dependent solutions γ_l^{\pm} ; they satisfy certain quadratic equations and have the forms

$$\begin{aligned} \gamma_l^+ = -\frac{d}{2} + \frac{1}{2} \sqrt{\frac{d^3 + 4ld^2 - d^2 - 4ld + 4l^2d - 8l + 4l^2}{(d-1)}}, \\ \gamma_l^- = -\frac{d}{2} + \frac{1}{2} \sqrt{\frac{d^3 + 4ld^2 - 9d^2 - 4ld + 4l^2d + 8 - 8l + 4l^2}{(d-1)}} \end{aligned} \quad (3.26)$$

(only one solution of each equation is admissible). Note that $\gamma_l^+ > \gamma_l^-$ for all l and $d > 1$. For large d , exponents (3.26) behave as $\gamma_l^+ = l + O(1/d)$ and $\gamma_l^- = l-2 + O(1/d)$, respectively, so that all solutions become “standard” at $d = \infty$.

Nonperturbative solutions for intermediate values of ε between 0 and 2 can only be obtained numerically. We have performed the calculation in two and three dimensions for $l \leq 12$; the results are illustrated by Fig. 2 for $l=2, 8$, and 12. For $d=3$ and $l \leq 10$, our solutions are in agreement with the results presented in Fig. 2 of Ref. [26], except for the case $l=2$: the behaviors of the solutions $\gamma_2 = 2 + O(\varepsilon)$ and $\gamma_2 = 4 + O(\varepsilon)$ are different. We believe that this disagreement is not conceptual and is explained by calculational errors in Ref. [26]. For the sake of brevity, we do not give the solutions for $l=4, 6$, and 10, which in three dimensions are in agreement with Ref. [26]. The exponent $\gamma=0$ for $l=2$ corresponds to the solution $[\delta_{ij} - dn_i n_j] \delta(\mathbf{p})$, which exists for all d (see Sec. III A).

The figures illustrate the following qualitative behavior of the solutions, which holds for all $d \neq 2$ and $l \geq 2$. The leading solution (3.21) exists for all ε and turns to $l-2$ for $\varepsilon=2$. In fact, it is hardly distinguishable from a constant, $\gamma_l \approx l-2$, for all values of ε ($\gamma_2 \equiv 0$, see Sec. III A).

For any l , some critical value $k_c = k_c(l, d)$ exists such that for all $k > k_c$ the behavior of the solutions $\gamma_l = l-2+2k + O(\varepsilon)$ is simple: the both solutions exist for all ε , decrease slowly as ε grows and turn to $\gamma_l = l-4+2k$ for $\varepsilon=2$. (In fact, the both solutions corresponding to given l and k are very close to each other for all values of ε .)

For fixed l , the critical value k_c decreases as d increases, so that all solutions with $k=1, 2, \dots$ become simple (in the above sense) provided d is large enough. For fixed d , the critical value k_c increases with l ; in particular, in three dimensions $k_c=2$ for $l=2, 4, 6$, $k_c=3$ for $l=8, 10$, and $k_c=4$ for $l=12$.

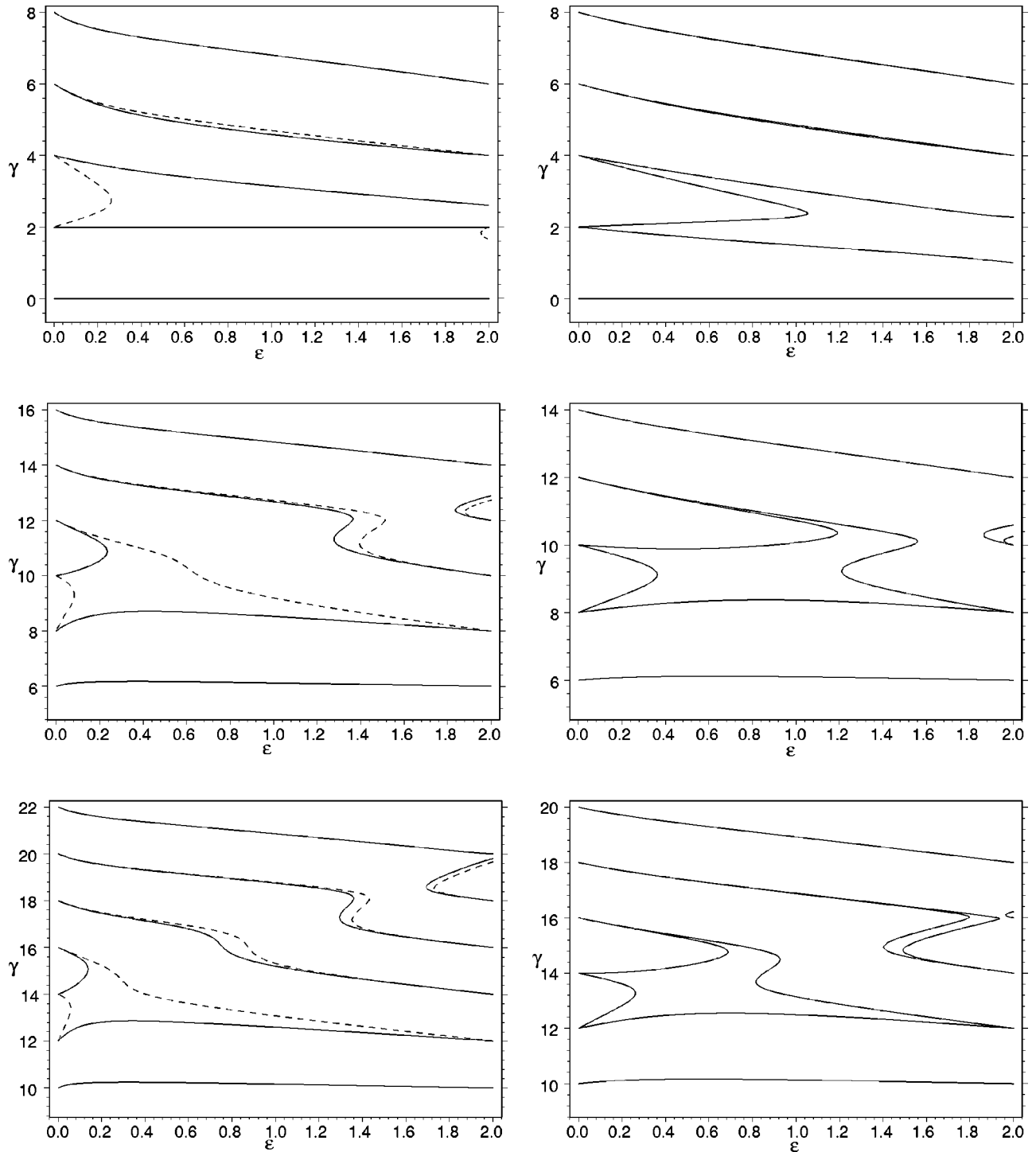


FIG. 2. Leading scaling exponents for the sectors $l=2, 8,$ and 12 (from above to below) in $d=2$ (left) and $d=3$ (right). Dashed lines denote solutions that exist as limits $d \rightarrow 2$ but disappear in two dimensions.

An interesting interaction between the solutions with a fixed l and different k occurs for $1 \leq k \leq k_c$. Two branches starting at $\epsilon=0$ with different values of k can coalesce and disappear for some value of ϵ between 0 and 2. Another possible process is the creation of a pair of solutions for some $0 < \epsilon < 2$. A solution that starts at $\epsilon=0$ can annihilate with a solution from a pair that was created for some finite value of ϵ . The interplay between these creation-annihilation

processes can produce a very complicated pattern, as illustrated by Fig. 2 for the sectors $l=2, 8,$ and 12 (see also Fig. 2 in Ref. [26]).

It turns out, however, that the creation and annihilation of solutions eventually compensate each other in the sense that the number of branches starting at $\epsilon=0$ with $0 < k < k_c$ is equal to the number of branches arriving at $\epsilon=2$ and confined between the leading solution ($\gamma_l = l-2$ for $\epsilon=2$) and

the lowest “simple” solution ($\gamma_l = l - 4 + 2k_c$ for $\varepsilon = 2$). The balance is possible owing to the existence of two “nonstandard” solutions (3.26) at the edge $\varepsilon = 2$. They also determine the boundary between the solutions with “simple” and “interesting” behavior: the uppermost solution with the interesting behavior, $\gamma_l = l - 2 + 2k_c + O(\varepsilon)$, turns to γ_l^+ at $\varepsilon = 2$. (Some reservations are needed if a “standard” solution at $\varepsilon = 2$ lies between the roots γ_l^\pm or coincides with one of them. In particular, for $d = 3$ and $l = 2$, the standard solution $\gamma_2 = 2$ exists but is isolated in the sense that no *real* branches attach it from the region $\varepsilon < 2$. For $d = 3$ and $l = 12$, one obtains $\gamma_{12}^- = 16$, and this standard value acquires threefold degeneracy).

The behavior eventually simplifies in the limit $d \rightarrow \infty$. All the solutions become simple in the above sense and they are described by straight lines: $\gamma_l = l - 2$ for the leading solution and $\gamma_l = l - 2 + 2k - \varepsilon$ for all $k \geq 1$.

The annihilation of coalescing solutions actually means that they become complex: the effect known for the magnetic model [27], where it occurs in the isotropic shell. It was argued in Refs. [27,29] that the complexification leads to the instability of the steady state (exponential growth of the pair correlation function). We shall not discuss this important issue here and only stress an essential distinction between the two cases. In the magnetic model, the *leading admissible* exponent $\gamma = O(\varepsilon)$ coalesces with the solution $\gamma = -d + O(\varepsilon)$, which is not admissible and describes the large-scale behavior at $r \gg L$ [see the remark and references below Eq. (3.5)]. In models (1.1)–(1.3) the coalescence occurs only in anisotropic sectors and only for *nonleading admissible* exponents. If the steady state remains stable, the inertial-range behavior in the corresponding sectors will include oscillations on the powerlike background; compare the discussion in Ref. [13].

In two dimensions, the tensor structures in decomposition (3.16) become coincident, and the determinant $\det[C_{\alpha\beta}^{(l)}]$ in Eq. (3.20) vanishes identically. All the coefficients $C_{\alpha\beta}^{(l)}$ for $d = 2$ become equal up to the sign [see Eq. (3.19) for $l = 2$]. Therefore, the equation for the exponents γ_l can simply be written as

$$C_{11}^{(l)} = 0. \quad (3.27)$$

All solutions of the d -dimensional equation (3.20) have well-defined limits as $d \rightarrow 2$, and all true two-dimensional solutions are indeed recovered in this limit. However, this limit gives more solutions than the correct two-dimensional equation: one-half of the solutions obtained in the limit $d \rightarrow 2$ from the d -dimensional case do not satisfy Eq. (3.27) and should be discarded. This behavior is illustrated by Fig. 2, where the solid lines on the diagrams with $d = 2$ and $l \geq 2$ denote solutions obtained both from the two-dimensional equation (3.27) and as limits $d \rightarrow 2$ from d -dimensional solutions, and the dashed lines denote spurious solutions that are obtained in the limit $d \rightarrow 2$ from Eq. (3.20), but do not satisfy equation (3.27). We shall see in Sec. VI C and VI D that similar effect is encountered in the RG and OPE approach: all critical dimensions have well-defined limits as d

TABLE I. Canonical dimensions of the fields and parameters in the model (2.1).

F	θ	θ'	\mathbf{v}	ν, ν_0	m, μ, Λ	g_0	g
d_F^k	0	d	-1	-2	1	ε	0
d_F^ω	-1/2	1/2	1	1	0	0	0
d_F	-1	$d+1$	1	0	1	ε	0

$\rightarrow 2$, but some of them should be ruled out due to linear relations between composite operators that hold in two dimensions.

IV. RENORMALIZATION, RG FUNCTIONS, AND RG EQUATIONS

The analysis of the UV divergences is based on the analysis of canonical dimensions [37,38]. Dynamical models of the type (2.1), in contrast to static models, have two scales, so that the canonical dimension of some quantity F (a field or a parameter in the action functional) is described by two numbers, the momentum dimension d_F^k and the frequency dimension d_F^ω . They are determined such that $[F] \sim [L]^{-d_F^k} [T]^{-d_F^\omega}$, where L is the length scale and T is the time scale. The dimensions are found from the obvious normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω , one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_t \propto \Delta$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems [38].

The dimensions for the model (2.1) are given in Table I, including the parameters that will be introduced later on. From the table it follows that the model is logarithmic (the coupling constant g_0 is dimensionless) at $\varepsilon = 0$, so that the UV divergences have the form of the poles in ε in the Green functions.

The total canonical dimension of an arbitrary one-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_{1-\text{ir}}$ is given by the relation

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi, \quad (4.1)$$

where $N_\Phi = \{N_{\theta'}, N_\theta, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ , and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose elimination requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences should be based on the following auxiliary considerations [16,17].

(i) From the explicit form of the vertex and bare propagators in the model (2.1) it follows that $N_{\theta'} - N_\theta = 2N_0$ for any one-irreducible Green function, where $N_0 \geq 0$ is the total

number of bare propagators $\langle \theta\theta \rangle_0$ entering into the function (see Sec. II). Therefore, the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any nonvanishing function.

(ii) If for some reason a number of external momenta occurs as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number (the Green function requires counterterms only if d'_Γ is a non-negative integer). In the model (2.1), the derivative ∂ at the vertex $\theta'(\mathbf{v}\partial)\theta$ can be moved onto the field θ' by virtue of the transversality of the field \mathbf{v} . Therefore, in any one-irreducible diagram it is always possible to move the derivative onto any of the external “tails” θ or θ' , which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_\theta - N_{\theta'}$. The fields θ , θ' enter into the counterterms only in the form of derivatives $\partial\theta$, $\partial\theta'$.

From the dimensions in Table I we find $d_\Gamma = d + 2 - N_\mathbf{v} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)(1 - N_{\theta'}) - N_\mathbf{v}$. It then follows that for any d , superficial divergences can only exist in the one-irreducible functions $\langle \theta' \theta \dots \theta \rangle_{1-\text{ir}}$ with $N_{\theta'} = 1$ and arbitrary value of N_θ , for which $d_\Gamma = 2$, $d'_\Gamma = 0$. However, all the functions with $N_\theta > N_{\theta'}$ vanish (see above) and obviously do not require counterterms. We are left with the only superficially divergent function $\langle \theta' \theta \rangle_{1-\text{ir}}$, which does not depend on the correlation function (1.2) and therefore is isotropic; see Sec. II. The corresponding counterterm must contain two symbols ∂ , and owing to the isotropy and transversality conditions reduces to the only structure $\theta' \Delta \theta$.

Inclusion of this counterterm is reproduced by the multiplicative renormalization of the parameters g_0 , ν_0 in the action functional (2.1) with the only independent renormalization constant Z_ν :

$$\nu_0 = \nu Z_\nu, \quad g_0 = g \mu^\varepsilon Z_g, \quad Z_g = Z_\nu^{-1}. \quad (4.2)$$

Here μ is the reference mass in the minimal subtraction (MS) scheme, which is always used in what follows, g and ν are renormalized analogs of the bare parameters g_0 and ν_0 , and $Z = Z(g, \varepsilon, d)$ are the renormalization constants. Their relation in Eq. (4.2) results from the absence of renormalization of the last term in Eq. (2.1). No renormalization of the fields and the “mass” m is required, i.e., $Z_\Phi = 1$ for all Φ and $m_0 = m$. The renormalized action functional has the form

$$S_R(\Phi) = \theta' D_\theta \theta' / 2 + \theta' [-\nabla_t + \nu Z_\nu \Delta] \theta - \mathbf{v} D_\nu^{-1} \mathbf{v} / 2, \quad (4.3)$$

where the amplitude D_0 from Eq. (1.3) expressed in renormalized parameters using Eqs. (4.2): $D_0 \equiv g_0 \nu_0 = g \mu^\varepsilon \nu$.

The explicit form of the constant Z_ν is determined by the requirement that the one-irreducible function $\langle \theta' \theta \rangle_{1-\text{ir}}$ expressed in renormalized variables be UV finite (i.e., be finite for $\varepsilon \rightarrow 0$). This requirement determines Z_ν up to an UV finite contribution; the latter is fixed by the choice of the renormalization scheme. In the MS scheme all renormalization constants have the form “1 + only poles in ε .” The function $\langle \theta' \theta \rangle_{1-\text{ir}}$ in our model is known exactly; see Eqs. (2.9)–(2.14) in Sec. II. We substitute Eqs. (4.2) into it and choose Z_ν to cancel the pole in ε in the resulting expression.

This is equivalent to the requirement that $\nu_{\text{eff}}(p)$ be finite; its pole part is independent of p and is therefore contained in Eq. (2.14). This gives

$$Z_\nu = 1 - g C_d \frac{(d^2 - 3)}{2d(d+2)\varepsilon}, \quad (4.4)$$

with coefficient C_d from Eq. (2.11). The result (4.4) is exact, i.e., it has no corrections of order g^2 , g^3 , and so on; this is a consequence of the fact that the one-loop approximation (2.7a) for the response function is exact. Also note that expression (4.4) differs from the exact expression for Z_ν in the scalar [8] and magnetic [23] cases.

The relation $\mathcal{S}(\Phi, e_0) = \mathcal{S}_R(\Phi, e, \mu)$ (where $e_0 = \{g_0, \nu_0, m\}$ is the complete set of bare parameters, and $e = \{g, \nu, m\}$ is the set of their renormalized analogs) implies $W(A, e_0) = W_R(A, e, \mu)$, where W is the functional (2.2) and W_R is its renormalized counterpart obtained by the replacement $\mathcal{S} \rightarrow \mathcal{S}_R$. We use $\tilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu \partial_\mu$ for fixed e_0 and operate on both sides of this relation with it. This gives the basic RG differential equation

$$\mathcal{D}_{RG} W_R(e, \mu) = 0, \quad (4.5)$$

where \mathcal{D}_{RG} is the operation $\tilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta(g) \partial_g - \gamma_\nu(g) \mathcal{D}_\nu. \quad (4.6)$$

In Eq. (4.6), we have written $\mathcal{D}_x \equiv x \partial_x$ for any variable x , and the RG functions (the β function and the anomalous dimension γ) are defined as

$$\gamma_F(g) \equiv \tilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any } Z_F, \quad (4.7a)$$

$$\beta(g) \equiv \tilde{\mathcal{D}}_\mu g = g[-\varepsilon + \gamma_\nu(g)]. \quad (4.7b)$$

The relation between β and γ in Eq. (4.7b) results from the definitions and the last relation in Eq. (4.2). From the relations (4.4) and (4.7) one obtains explicit expressions for the RG functions:

$$\gamma_\nu(g) = \frac{-\varepsilon \mathcal{D}_g \ln Z_\nu}{1 - \mathcal{D}_g \ln Z_\nu} = g C_d \frac{(d^2 - 3)}{2d(d+2)}. \quad (4.8)$$

From Eq. (4.7b) it follows that the RG equations of the model have an IR stable fixed point [$\beta(g_*) = 0$, $\beta'(g_*) > 0$] with the coordinate

$$g_* = \frac{2d(d+2)\varepsilon}{C_d(d^2-3)}. \quad (4.9)$$

From the relation between the RG functions in Eq. (4.7b) the value of $\gamma_\nu(g)$ at the fixed point is found exactly: $\gamma_\nu^* \equiv \gamma_\nu(g_*) = \varepsilon$.

For $d^2 < 3$, the fixed point is negative and therefore not accessible for the RG flow with physical (positive) initial data for g . This is in agreement with the conclusion of Sec. II

that no stable steady state exists in the model for $d^2 < 3$ [see the discussion below Eq. (2.16)].

For $d^2 > 3$, the fixed point is positive; this establishes the existence of scaling behavior in the IR region ($\Lambda r \gg 1$ and any fixed mr) for all correlation functions of the model. Let F be some multiplicatively renormalized quantity (say, a correlation function involving composite operators), i.e., $F = Z_F F_R$ with certain renormalization constant Z_F . It satisfies the RG equation of the form $[\mathcal{D}_{RG} + \gamma_F]F_R = 0$ with γ_F from Eq. (4.7a) and \mathcal{D}_{RG} from Eq. (4.6). The solution of the RG equation then shows that in the IR region F takes on the scaling form

$$F \simeq \Lambda^{-\gamma_F^*} D_0^{\omega} r^{-\Delta_F} \xi_F(mr), \quad (4.10)$$

where

$$\Delta_F = d_F^k + \Delta_\omega d_F^\omega + \gamma_F^*, \quad \Delta_\omega = 2 - \gamma_\nu^* \quad (4.11)$$

is the critical dimension of the function F , d_F^ω , and d_F are its frequency and total canonical dimensions, $\gamma_F^* = \gamma_F(g_*)$ is the value of its anomalous dimension at the fixed point, $\Delta_\omega = 2 - \gamma_\nu^* = 2 - \varepsilon$ is the critical dimension of the frequency, and $\xi_F(mr)$ is the scaling function whose form is not determined by the RG equation itself. Derivation of Eq. (4.11) and more detail can be found in Refs. [14,16,17,38]. In particular, for the structure functions (1.6) with $d_F = 0$, $d_F^\omega = -n/2$ (see Table I) and $\gamma_F^* = 0$ (see Sec. VI) one obtains

$$S_n(\mathbf{r}) = D_0^{-n/2} r^{n(1-\varepsilon/2)} \xi_n(mr), \quad (4.12)$$

so that the dependence on the UV scale Λ disappears, while the dependence on the IR scale m is contained in the scaling functions $\xi_n(mr)$.

V. OPERATOR PRODUCT EXPANSION AND ANOMALOUS SCALING

Representations (4.10)–(4.12) for any scaling functions $\xi(mr)$ describe the behavior of the correlation functions for $\Lambda r \gg 1$ and any fixed value of mr . The inertial range corresponds to the additional condition $mr \ll 1$. The form of the functions $\xi(mr)$ is not determined by the RG equations themselves; in analogy with the theory of critical phenomena, their behavior for $mr \rightarrow 0$ is studied using OPE; see Refs. [37,38]. Below we concentrate on the equal-time structure functions (1.6) and (4.12).

According to the OPE, the behavior of the quantities entering into the right-hand side of Eq. (1.6) for $\mathbf{r} = \mathbf{x} - \mathbf{x}' \rightarrow 0$ and fixed $\mathbf{x} + \mathbf{x}'$ is given by the infinite sum

$$[\theta_r(t, \mathbf{x}) - \theta_r(t, \mathbf{x}')]^n = \sum_F C_F(\mathbf{r}) F\left(t, \frac{\mathbf{x} + \mathbf{x}'}{2}\right), \quad (5.1)$$

where C_F are coefficients regular in m^2 and F are all possible renormalized local composite operators allowed by the symmetry. More precisely, the operators entering into the OPE are those that appear in the naive Taylor expansion, and all the operators that admit to them in renormalization.

In what follows it is assumed that the expansion is made in irreducible tensors (scalars, vectors, and traceless tensors); the possible tensor indices of the operators F are contracted with the corresponding indices of the coefficients C_F . With no loss of generality, it can also be assumed that the expansion is made in ‘‘scaling’’ operators, i.e., those having definite critical dimensions Δ_F (see Sec. VI).

The structure functions (1.6) are obtained by averaging Eq. (5.1) with the weight $\exp \mathcal{S}_R$, the mean values $\langle F \rangle$ appear on the right-hand side. Their asymptotic behavior for $m \rightarrow 0$ is found from the corresponding RG equations and has the form $\langle F \rangle \propto m^{\Delta_F}$.

From the RG representation (4.12) and the operator product expansion (5.1) we therefore find the following expression for the structure function in the inertial range ($\Lambda r \gg 1$, $mr \ll 1$)

$$S_n(\mathbf{r}) = D_0^{-n/2} r^{n(1-\varepsilon/2)} \sum_F A_F(m\mathbf{r})(mr)^{\Delta_F}, \quad (5.2)$$

where the coefficients A_F are regular in $(mr)^2$.

Some general remarks are now in order.

Owing to the translational invariance, the operators having the form of total derivatives give no contribution to Eq. (5.2): $\langle \partial F(x) \rangle = \partial \langle F(x) \rangle = \partial \times \text{const} = 0$ (these operators become relevant if the stirring force in Eq. (1.1) violates the translational invariance, like in the problem discussed in Ref. [40]).

In the models (1.1)–(1.3), the operators with an odd number of fields θ also have vanishing mean values; their contributions vanish along with the odd structure functions themselves (they will be ‘‘activated’’ in the presence of a nonzero mixed correlation function $\langle \mathbf{v}f \rangle$; we shall not discuss this possibility here).

If the tensor $C_{ij}(\mathbf{r})$ in Eq. (1.2) is taken to be isotropic, the model becomes $\text{SO}(d)$ covariant and only the contributions of the scalar operators survive in Eq. (5.2). Indeed, in the isotropic case the mean value of a tensor operator depends only on scalar parameters, its tensor indices can only be those of Kronecker δ symbols. It is impossible, however, to construct nonzero irreducible (traceless) tensor solely of the δ symbols.

In the presence of anisotropy, irreducible tensor operators acquire nonzero mean values and their contributions appear on the right-hand side of Eq. (5.2). Like in Sec. III, consider the case of the uniaxial anisotropy, specified by a unit vector \mathbf{n} in the correlation function (1.2). In this case, the mean value of a l th rank traceless operator involves the vector \mathbf{n} along with the δ symbols and is necessarily proportional to the l th rank symmetric traceless tensor $\mathcal{N}_{i_1 \dots i_l}^{(l)}$ from Eq. (3.11). The contraction with the corresponding coefficient C_F gives rise to the l th order Gegenbauer polynomial $P_l(z)$ with $z = (\mathbf{r}\mathbf{n})/r$; see Eq. (3.13). In general, the expansion in irreducible tensors in Eq. (5.1) after the averaging leads to the $\text{SO}(d)$ decomposition employed in Refs. [26,30], the l th shell corresponding to the contribution of the l th rank composite operators.

The feature characteristic of the models describing turbulence is the existence of the so-called ‘‘dangerous’’ composite operators with *negative* critical dimensions; see Refs. [16,17]. Their contributions into the OPE give rise to singular behavior of the scaling functions for $mr \rightarrow 0$, that is, the anomalous scaling. The leading term in the l th anisotropic sector is given by the l th rank tensor operator with minimal (not necessarily negative) dimension $\Delta[F]$.

Since $\Delta_F = d_F + O(\varepsilon)$, see Eq. (4.11), the operators with minimal Δ_F are those involving maximum possible number of fields θ and minimum possible number of derivatives (at least for small ε). Both the problem (1.1)–(1.3) and the quantities (1.6) possess the symmetry $\theta \rightarrow \theta + \text{const}$. It then follows that the expansion (5.1) involves only operators invariant with respect to this shift and therefore built of the *gradients* of θ .

As already mentioned above, the operators entering into the right-hand side of Eq. (5.1) are those that appear in the Taylor expansion, and those that admix to them in renormalization. The leading term of the Taylor expansion for S_n is the $2n$ th rank operator that can symbolically be written as $(\partial\theta)^n$; its decomposition in irreducible tensors gives rise to operators of lower ranks. These contributions exist in the OPE (before averaging) even if the stirring force is not included into Eq. (1.1); in the language of Refs. [4–6] it is then tempting to identify them with zero modes, i.e., the solutions of the homogeneous (unforced) analogs of the closed exact equations satisfied by the equal-time correlations. In the presence of the stirring force, operators of the form $(\partial\theta)^k$ with $k < n$ admix to them in renormalization and appear in the OPE; their contributions correspond to solutions of the inhomogeneous equations. Owing to the linearity of problem (1.1), operators with $k > n$ (whose contributions would be more important) do not admix in renormalization to the terms of the Taylor expansion for S_n and do not appear in the corresponding OPE. All these operators have minimal possible canonical dimension $d_F = 0$ (see Table I) and determine the leading terms of the $mr \rightarrow 0$ behavior in the sectors with $j \leq 2n$. Operators involving more derivatives than fields θ (and thus having canonical dimensions $d_F = 1, 2$ and so on) determine correction terms for $j \leq 2n$ and leading terms for higher anisotropic sectors with $j > 2n$. The renormalization and dimensions of the most important operators are studied in the following section.

VI. RENORMALIZATION AND CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

We recall that the term ‘‘composite operator’’ refers to any local (unless stated to be otherwise) monomial or polynomial built of primary fields and their derivatives at a single space-time point $x \equiv (t, \mathbf{x})$; see Refs. [37,38]. Since the arguments of the fields coincide, correlation functions with such operators contain additional UV divergences, which are removed by additional renormalization procedure. For the renormalized correlation functions the RG equations are obtained, which describe IR scaling of certain ‘‘basis’’ operators F with definite critical dimensions $\Delta_F \equiv \Delta[F]$. Due to the renormalization, $\Delta[F]$ does not coincide in general with

the naive sum of critical dimensions of the fields and derivatives entering into F . As a rule, composite operators ‘‘mix’’ in renormalization, i.e., an UV finite renormalized operator F^R has the form $F^R = F + \text{counterterms}$, where the contribution of the counterterms is a linear combination of F itself and, possibly, other unrenormalized operators that ‘‘admix’’ to F .

Let $F \equiv \{F_{\alpha j}\}$ be a closed set, all of whose monomials mix only with each other in renormalization. The renormalization matrix $Z_F \equiv \{Z_{\alpha\beta}\}$ and the matrix of anomalous dimensions $\gamma_F \equiv \{\gamma_{\alpha\beta}\}$ for this set are given by

$$F_{\alpha} = \sum_{\beta} Z_{\alpha\beta} F_{\beta}^R, \quad \gamma_F = Z_F^{-1} \tilde{D}_{\mu} Z_F, \quad (6.1)$$

and the corresponding matrix of critical dimensions $\Delta_F \equiv \{\Delta_{\alpha\beta}\}$ is given by Eq. (4.11), in which d_F^k and d_F^{ω} are understood as the diagonal matrices of canonical dimensions of the operators in question (with the diagonal elements equal to sums of corresponding dimensions of all fields and derivatives constituting F) and $\gamma_F^* \equiv \gamma_F(g_*)$ is the matrix (6.1) at the fixed point (4.9).

Critical dimensions of the set $F \equiv \{F_{\alpha}\}$ are given by the eigenvalues of the matrix Δ_F . The ‘‘basis’’ operators that possess definite critical dimensions have the form

$$F_{\alpha}^{bas} = \sum_{\beta} U_{\alpha\beta} F_{\beta}^R, \quad (6.2)$$

where the matrix $U_F = \{U_{\alpha\beta}\}$ is such that $\Delta'_F = U_F \Delta_F U_F^{-1}$ is diagonal.

In general, counterterms to a given operator F are determined by all possible one-irreducible Green functions with one operator F and arbitrary number of primary fields, $\Gamma = \langle F(x) \Phi(x_1) \dots \Phi(x_2) \rangle_{1-\text{ir}}$. The total canonical dimension (formal index of UV divergence) for such a function is given by

$$d_{\Gamma} = d_F - N_{\Phi} d_{\Phi}, \quad (6.3)$$

with the summation over all types of fields entering into the function. For superficially divergent diagrams, d_{Γ} is a non-negative integer; cf. Sec. IV.

Let us begin with the simplest operators of the form $\theta^n(x)$, with free tensor indices or involving any contraction. From Table I in Sec. IV and Eq. (6.3) we obtain $d_F = -n$, $d_{\Gamma} = -n + N_{\theta} - N_{\nabla} - (d+1)N_{\theta'}$.

From the analysis of the diagrams it follows that the total number N_{θ} of the fields θ entering into the one-irreducible function $\Gamma = \langle \theta^n(x) \theta(x_1) \dots \theta(x_{N_{\theta}}) \rangle_{1-\text{ir}}$ cannot exceed the number of the fields θ in the operator θ^n itself, i.e., $N_{\theta} \leq n$ [cf. item (i) in Sec. IV]. Therefore, the divergence can only exist in the functions with $N_{\nabla} = N_{\theta'} = 0$, and arbitrary value of $n = N_{\theta}$, for which the formal index vanishes, $d_{\Gamma} = 0$. However, at least one of N_{θ} external ‘‘tails’’ of the field θ is attached to a vertex $\theta'(\mathbf{v}\partial)\theta$ (it is impossible to construct nontrivial, superficially divergent diagram of the desired type with all the external tails attached to the vertex F), at least

one derivative ∂ appears as an extra factor in the diagram, and, consequently, the real index of divergence d'_1 is necessarily negative.

This means that the operators θ^n require no counterterms at all, i.e., they are in fact UV finite, $\theta^n = Z[\theta^n]^R$ with $Z = 1$. It then follows that the critical dimension of $\theta^n(x)$ is simply given by the expression (4.11) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors: $\Delta[\theta^n] = n\Delta[\theta] = n(-1 + \varepsilon/2)$. Since the structure functions (1.6) are linear combinations of pair correlation function involving the operators θ^n , this relation shows that they indeed satisfy the homogeneous RG equation (4.5), discussed in Sec. IV.

In the OPE for the pair correlation function, analogous to Eq. (5.2), the operators θ^2 and $\theta_i\theta_j$ with the dimensions $2(-1 + \varepsilon/2)$ give rise to constant terms. They correspond to the solutions with $\delta(\mathbf{p})$, discussed in the end of Sec. III. Such terms, caused by various operators of the form θ^n , are also present in higher-order correlation functions. They disappear from the structure functions (1.6), whose inertial-range behavior is determined by operators built only of gradients (see Sec. V).

A. Scalar operators of the form $(\partial\theta)^2$ and the scaling of S_2

The leading terms of the inertial-range behavior of the second-order structure function S_2 are determined by the critical dimensions of the composite operators built of two gradients:

$$F_1 = \partial_i\theta_j\partial_j\theta_i, \quad F_2 = \partial_i\theta_j\partial_j\theta_i. \quad (6.4)$$

For the transverse field θ , the second operator reduces to a total derivative, $F_2 = \partial_i\partial_j(\theta_j\theta_i)$, and its dimension $\Delta_2 = 2 + 2\Delta_\theta = \varepsilon$ does not appear on the right-hand side of Eq. (5.2).

The dimension of the first operator is found exactly: $\Delta_1 = 0$. This can be demonstrated using the Schwinger equation of the form

$$\int \mathcal{D}\Phi \frac{\delta}{\delta\theta'_i(x)} \{ \theta_i(x) \exp[S_R(\Phi) + A\Phi] \} = 0 \quad (6.5)$$

(in the general sense of the term, Schwinger equations are any relations stating that any functional integral of a total variational derivative is equal to zero, see, e.g., Refs. [37,38]). Here S_R is the renormalized analog of the action (2.1), and the notation introduced in Eq. (2.2) is used. Equation (6.5) can be rewritten in the form

$$\begin{aligned} & \langle \langle \theta' D_\theta \theta - \nabla_i[\theta^2/2] - \partial_i[\theta_i\mathcal{P}] + \nu Z_\nu \Delta[\theta^2/2] - \nu Z_\nu F_1 \rangle \rangle_A \\ & = -A_{\theta'} \delta W_R(A) / \delta A_\theta. \end{aligned} \quad (6.6)$$

Here D_θ is the correlation function (1.2), $\theta^2 \equiv \theta_i\theta_i$, $\langle \langle \dots \rangle \rangle_A$ denotes the averaging with the weight $\exp[S_R(\Phi) + A\Phi]$, W_R is determined by Eq. (2.2) with the replacement $\mathcal{S} \rightarrow S_R$, and the argument x common to all the quantities is omitted. The contribution with the pressure \mathcal{P}

from Eq. (1.5) arises due to the fact that the differentiation in Eq. (6.5) is performed with respect to a transverse field; see the remark below Eq. (2.1).

The quantity $\langle \langle F \rangle \rangle_A$ is the generating functional of the correlation functions with one operator F and any number of fields Φ , therefore the UV finiteness of the operator F is equivalent to the finiteness of the functional $\langle \langle F \rangle \rangle_A$. The quantity on the right-hand side of Eq. (6.6) is finite (a derivative of the renormalized functional with respect to a finite argument), and so is the operator on the left-hand side. Our operator F_1 does not admit renormalization to the operator $\theta' D_\theta \theta$ (F_1 contains too many fields θ), and to the operators $\nabla_i[\theta^2/2]$, $\partial_i[\theta_i\mathcal{P}]$ and $\Delta[\theta^2/2]$ (they have the form of total derivatives, and F_1 does not reduce to this form). On the other hand, the operators $\theta' D_\theta \theta$ and $\partial_i[\theta_i\mathcal{P}]$ do not admit to F_1 (they are nonlocal, and F_1 is local), while the derivatives $\nabla_i[\theta^2/2]$ and $\Delta[\theta^2/2]$ do not admit to F_1 owing to the fact that each field θ enters into the counterterms of the operators F_n only in the form of derivative $\partial\theta$ (see above). Therefore, all three types of operators entering into the left-hand side of Eq. (6.6) are independent, and they must be UV finite separately.

Since the operator $\nu Z_\nu F_1$ is UV finite, it coincides with its finite part, i.e., $\nu Z_\nu F_1 = \nu F_1^R$, which along with the relation $F_1 = Z_1 F_1^R$ gives $Z_1 = Z_\nu^{-1}$ and therefore $\gamma_1 = -\gamma_\nu$. For the critical exponent $\Delta_1 = \varepsilon + \gamma_1^*$ we then obtain $\Delta_1 = 0$ exactly (we recall that $\gamma_\nu^* = \varepsilon$; see the discussion below Eq. (4.9) in Sec. IV).

It then follows from Eq. (5.2) that the leading term of the inertial-range behavior of the second-order structure function has the form $S_2 \propto D_0^{-1} r^{2-\varepsilon}$, in agreement with the solution $\gamma = 2 - \varepsilon$ obtained in Sec. III from the exact equation. Therefore, this function is not anomalous, like its analog for the scalar model [3–5], and the anomalous scaling reveals itself only on the level of the fourth-order structure function.

B. Scalar operators of the form $(\partial\theta)^4$ and the anomalous scaling of S_4

Let us turn to the scalar composite operators built of four gradients $\partial\theta$, which cannot be reduced to the form of total derivatives. This family includes six independent monomials, all of which can be obtained from the fourth rank operator $\Phi_{ijkl}^{mnpq} \equiv \partial_i\theta_m\partial_j\theta_n\partial_k\theta_p\partial_l\theta_s$ by various contractions of the tensor indices:

$$\begin{aligned} F_1 &= \Phi_{jilk}^{ijkl}, & F_2 &= \Phi_{jjll}^{iikk}, & F_3 &= \Phi_{jill}^{ijkk}, & F_4 &= \Phi_{jkll}^{iijk}, \\ F_5 &= \Phi_{klkl}^{ijij}, & F_6 &= \Phi_{jklk}^{iijk}. \end{aligned} \quad (6.7)$$

At first glance, it seems that one can add another independent monomial, $F_7 = \Phi_{lijk}^{ijkl}$, but in fact it reduces to F_1 up to total derivatives

$$3F_1 - 6F_7 = \partial_i[-6\theta_k\Phi_{ksp}^{spi} + 3\theta_k\Phi_{skp}^{pis} + 2\theta_l\Phi_{skpl}^{kps}], \quad (6.8)$$

where the notation is analogous to that in Eq. (6.7); see Ref. [41].

Now let us turn to the calculation of the renormalization constants for the family (6.7) in the one-loop approximation. Let $\Gamma_\alpha(x; \theta)$ be the generating functional of the one-irreducible Green functions with one composite operator F_α from Eq. (6.7) and any number of fields θ . Here $x \equiv (t, \mathbf{x})$ is the argument of the operator and θ is the functional argument, the ‘‘classical counterpart’’ of the random field θ . We are interested in the fourth term of the expansion of $\Gamma_\alpha(x; \theta)$ in θ , which we denote $\Gamma_\alpha^{(4)}(x; \theta)$. It has the form

$$\Gamma_\alpha^{(4)}(x; \theta) = \frac{1}{4!} \int dx_1 \cdots \int dx_4 \theta_{i_1}(x_1) \cdots \theta_{i_4}(x_4) \times \langle F_\alpha(x) \theta_{i_1}(x_1) \cdots \theta_{i_4}(x_4) \rangle_{1-\text{ir}}.$$

In the one-loop approximation this function is represented diagrammatically as follows:

$$\Gamma_\alpha^{(4)}(x; \theta) = F_\alpha + \frac{1}{2} \text{Diagram} \quad (6.9)$$

Here the thin solid lines denote the *bare* propagator $\langle \theta \theta' \rangle_0$ from Eq. (2.4a), the ends with a slash correspond to the field θ' , and the ends without a slash correspond to θ ; the wavy line denotes the velocity propagator (1.3); the vertices correspond to the factor (2.3). The first term is the ‘‘tree’’ approximation, and the black circle with two attached lines in the diagram denotes the variational derivative

$$V_{i_1 i_2}^{(\alpha)}(x; x_1, x_2) \equiv \delta^2 F_\alpha(x) / \delta \theta_{i_1}(x_1) \delta \theta_{i_2}(x_2). \quad (6.10)$$

The diagram is written analytically in the form

$$\int dx_1 \cdots \int dx_4 V_{i_1 i_2}^{(\alpha)}(x; x_1, x_2) \times \langle \theta_{i_1}(x_1) \theta'_{i_3}(x_3) \rangle_0 \langle \theta_{i_2}(x_2) \theta'_{i_4}(x_4) \rangle_0 \times \langle v_{i_5}(x_3) v_{i_6}(x_4) \rangle_0 \partial_{i_5} \theta_{i_3}(x_3) \partial_{i_6} \theta_{i_4}(x_4), \quad (6.11)$$

with the bare propagators from Eqs. (1.3) and (2.4); the derivatives appear from the ordinary vertices (2.3). It is convenient to represent the vertex (6.10) in the form

$$V_{i_1 i_2}^{(\alpha)}(x; x_1, x_2) = \frac{\partial^2 F_\alpha(a)}{\partial a_{i_3 i_4} \partial a_{i_5 i_6}} \partial_{i_1} \delta_{i_3 i_4}(x - x_1) \partial_{i_2} \delta_{i_5 i_6}(x - x_2), \quad (6.12)$$

where

$$\delta_{ij}(x - x') \equiv \delta(t - t') P_{ij}(\mathbf{x} - \mathbf{x}') = \delta(t - t') \times \int \mathcal{D}\mathbf{k} P_{ij}(\mathbf{k}) \exp[\mathbf{i}\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]$$

is the δ function on the transverse subspace. The first (combinatorial) factor in Eq. (6.12) is understood as follows: the gradients $\partial_i \theta_j(x)$ in the operator F_α are replaced with a constant tensor a_{ij} , the differentiation is performed with respect

to its elements, and *after the differentiation* they are replaced back with the gradients, $a_{ij} \rightarrow \partial_i \theta_j(x)$.

Using the identity $\partial_k \delta_{ij}(x - x') = -\partial'_k \delta_{ij}(x - x')$ and the integration by parts, the derivatives can be moved from the vertex onto the propagators, and the integrations with respect to x_1 and x_2 are then easily performed:

$$\frac{\partial^2 F_\alpha(a)}{\partial a_{i_7 i_1} \partial a_{i_8 i_2}} \int dx_3 \int dx_4 \langle \partial_{i_7} \theta_{i_1}(x) \theta'_{i_3}(x_3) \rangle_0 \langle \partial_{i_8} \theta_{i_2}(x) \theta'_{i_4}(x_4) \rangle_0 \langle v_{i_5}(x_3) v_{i_6}(x_4) \rangle_0 \partial_{i_5} \theta_{i_3}(x_3) \partial_{i_6} \theta_{i_4}(x_4). \quad (6.13)$$

In order to find the renormalization constants, we do not need the entire exact expression (6.13), rather we need its UV divergent part. The latter is proportional to a polynomial built of four factors $\partial \theta$ at a single space-time point x . The needed four gradients have already been factored out from the expression (6.11): two factors from the vertex (6.12) and two factors from the ordinary vertices (2.3). Therefore, we can neglect the space-time inhomogeneity of the gradients and replace them with their values at the point x . Expression (6.13) can therefore be written, up to an UV finite part, in the form

$$\frac{\partial^2 F_\alpha(a)}{\partial a_{i_7 i_1} \partial a_{i_8 i_2}} \partial_{i_5} \theta_{i_3}(x) \partial_{i_6} \theta_{i_4}(x) X_{i_7 i_1 i_3 i_8 i_2 i_4 i_5 i_6}, \quad (6.14)$$

where we have denoted

$$X_{i_7 i_1 i_3 i_8 i_2 i_4 i_5 i_6} \equiv \int dx_3 \int dx_4 \langle \partial_{i_7} \theta_{i_1}(x) \theta'_{i_3}(x_3) \rangle_0 \langle \partial_{i_8} \theta_{i_2}(x) \theta'_{i_4}(x_4) \rangle_0 \langle v_{i_5}(x_3) v_{i_6}(x_4) \rangle_0 \quad (6.15)$$

or, in the momentum-frequency representation, after the integration over the frequency,

$$X_{i_7 i_1 i_3 i_8 i_2 i_4 i_5 i_6} = \frac{D_0}{2\nu_0} \int \frac{\mathcal{D}\mathbf{k}}{k^{d+\varepsilon}} P_{i_1 i_3}(\mathbf{k}) P_{i_2 i_4}(\mathbf{k}) P_{i_5 i_6}(\mathbf{k}) \frac{k_{i_7} k_{i_8}}{k^2}, \quad (6.16)$$

with D_0 from Eq. (1.3). Using the isotropy relations

$$\int \mathcal{D}\mathbf{k} f(k) \frac{k_{i_1} \cdots k_{i_{2n}}}{k^{2n}} = \frac{\delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{2n-1} i_{2n}} + (\text{all possible permutations})}{d(d+2) \cdots (d+2n-2)} \times \int \mathcal{D}\mathbf{k} f(k) \quad (6.17)$$

the integral (6.16) can be reduced to the simple scalar integral

$$\int \frac{D\mathbf{k}}{k^{d+\varepsilon}} = C_d \frac{m^{-\varepsilon}}{\varepsilon}, \quad (6.18)$$

with C_d from Eq. (2.11); the parameter m has arisen from the lower limit in the integral over \mathbf{k} . The explicit answer for the quantity (6.16) is given in Appendix B.

Contraction of the δ symbols with the first factors in Eq. (6.14) gives rise to various monomials built of four gradients of θ ; up to total derivatives, they reduce to the operators from the family (6.7). Then the function $\Gamma_\alpha^{(4)}(x; \theta)$ from Eq. (6.9) in the one-loop approximation of the renormalized perturbation theory (i.e., to the first order in g) up to an UV finite part can be written in the form

$$\Gamma_\alpha^{(4)}(x; \theta) = F_\alpha + \frac{g C_d}{\varepsilon} (\mu/m)^\varepsilon R_\alpha, \quad (6.19)$$

where μ has appeared from the relation $D_0 = g_0 \nu_0 = g \nu \mu^\varepsilon$ and

$$R_\alpha = \sum_\beta A_{\alpha\beta} F_\beta \quad (6.20)$$

are linear combinations of the monomials (6.7) with the coefficients $A_{\alpha\beta}$ dependent only on d :

$$\begin{aligned} R_1 = & -\frac{(3d^2 + 22d + 36)F_1}{d(d+2)(d+4)(d+6)} + \frac{2(d+3)F_2}{d(d+2)(d+4)(d+6)} \\ & + \frac{6F_3}{d(d+2)(d+6)} + \frac{2(d+5)F_4}{(d+2)(d+6)} \\ & + \frac{4(d+3)F_5}{d(d+2)(d+4)(d+6)} - \frac{4F_6}{(d+2)(d+6)}, \end{aligned} \quad (6.21)$$

$$\begin{aligned} R_2 = & -\frac{12F_1}{d(d+2)(d+4)(d+6)} \\ & + \frac{(d^4 + 10d^3 + 19d^2 - 44d - 90)F_2}{d(d+2)(d+4)(d+6)} + \frac{(d+12)F_3}{d(d+2)(d+6)} \\ & + \frac{12F_4}{d(d+2)(d+6)} + \frac{2(d^3 + 8d^2 + 10d - 18)F_5}{d(d+2)(d+4)(d+6)} \\ & - \frac{4F_6}{(d+2)(d+6)}, \end{aligned}$$

$$\begin{aligned} R_3 = & \frac{(d^2 + 18d + 48)F_1}{2d(d+2)(d+4)(d+6)} + \frac{2(2d+9)F_2}{d(d+2)(d+4)(d+6)} \\ & + \frac{(d^2 + 4d - 15)F_3}{2d(d+6)} - \frac{2F_4}{(d+2)(d+6)} \\ & - \frac{2(d^2 + 6d + 6)F_5}{d(d+2)(d+4)(d+6)} + \frac{2(d+4)F_6}{(d+2)(d+6)}, \end{aligned}$$

$$\begin{aligned} R_4 = & \frac{2(d+3)F_1}{d(d+2)(d+4)(d+6)} + \frac{(d+3)(d^2 + 8d + 14)F_2}{2d(d+2)(d+4)(d+6)} \\ & + \frac{3F_3}{d(d+2)(d+6)} + \frac{(d^3 + 5d^2 - 14d - 36)F_4}{2d(d+2)(d+6)} \\ & - \frac{2(d+3)(d+5)F_5}{d(d+2)(d+4)(d+6)} - \frac{2F_6}{(d+2)(d+6)}, \\ R_5 = & -\frac{6F_1}{d(d+2)(d+4)(d+6)} + \frac{(d^3 + 10d^2 + 27d + 15)F_2}{d(d+2)(d+4)(d+6)} \\ & + \frac{3F_3}{d(d+2)(d+6)} - \frac{2(d+3)F_4}{d(d+2)(d+6)} \\ & + \frac{(d^4 + 10d^3 + 18d^2 - 54d - 114)F_5}{d(d+2)(d+4)(d+6)} \\ & + \frac{12F_6}{d(d+2)(d+6)}, \end{aligned}$$

$$\begin{aligned} R_6 = & \frac{(d^2 + 18d + 48)F_1}{4d(d+2)(d+4)(d+6)} + \frac{(2d+9)F_2}{d(d+2)(d+4)(d+6)} \\ & + \frac{(d^2 + 6d + 6)F_3}{2d(d+2)(d+6)} - \frac{F_4}{(d+2)(d+6)} \\ & - \frac{(d^2 + 6d + 6)F_5}{d(d+2)(d+4)(d+6)} + \frac{(d^3 + 6d^2 - 11d - 42)F_6}{2d(d+2)(d+6)}. \end{aligned}$$

The constants $Z_{\alpha\beta}$ are found from the requirement that the functions (6.9) for the renormalized analogs of operators (6.7), defined by the relation $F_\alpha = Z_{\alpha\beta} F_\alpha^R$, be UV finite, i.e., be finite for $\varepsilon \rightarrow 0$. In the MS scheme this gives

$$Z_{\alpha\beta} = \delta_{\alpha\beta} + g C_d A_{\alpha\beta} / \varepsilon + O(g^2) \quad (6.22)$$

with the coefficients $A_{\alpha\beta}$ from Eq. (6.21). For the matrix of anomalous dimensions (6.1) at the fixed point (4.9) one has $\gamma_{\alpha\beta}^* = -g_* C_d A_{\alpha\beta} + O(\varepsilon^2)$, and for the matrix of critical dimensions from Table I and Eq. (4.11) one obtains

$$\Delta_{\alpha\beta} = 2\varepsilon \delta_{\alpha\beta} + \gamma_{\alpha\beta}^*. \quad (6.23)$$

Critical dimensions associated with the family of operators (6.7) are given by the eigenvalues Δ_α of the matrix (6.23). In particular, for $d=3$ one obtains

$$\begin{aligned} \Delta_1 \approx & -0.55\varepsilon, \quad \Delta_2 \approx 0.68\varepsilon, \quad \Delta_3 \approx 1.1\varepsilon, \\ \Delta_4 \approx & 2.4\varepsilon, \quad \Delta_5 = 8\varepsilon/3, \quad \Delta_6 = 3\varepsilon. \end{aligned} \quad (6.24)$$

For general d , only one of the eigenvalues is found analytically,

$$\Delta_5 = \frac{(d+1)^2}{(d^2-3)} \varepsilon \quad (6.25)$$

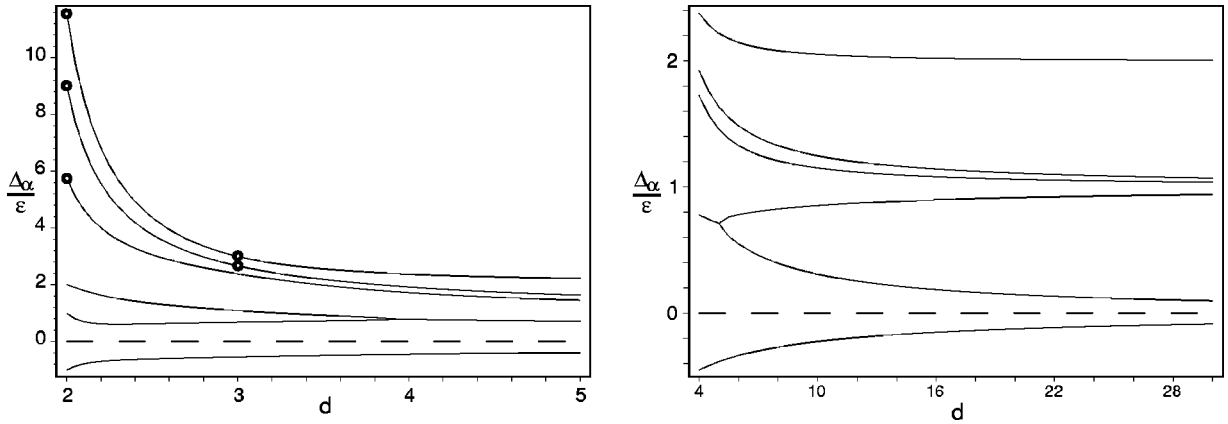


FIG. 3. Critical dimensions Δ_1 – Δ_6 (from below to above) of the operators (6.7) in the order $O(\varepsilon)$ vs the space dimensionality d for $2 \leq d \leq 5$ (left) and $4 \leq d \leq 30$ (right). The empty circles denote the operators which become trivial in $d=2$ and 3. The dimensions tend to 0 ($\Delta_{1,2}$), ε (Δ_3 – Δ_5), and 2ε (Δ_6) for $d \rightarrow \infty$.

[we recall that all these eigenvalues have corrections of order $O(\varepsilon^2)$ and higher]. The critical dimensions Δ_α as functions of d are presented in Fig. 3. They are always real, except for the pair $\Delta_{2,3}$ which becomes complex conjugate in the interval $4 < d < 5$ (in Fig. 3 the real part is shown). One can also see that for all d , exactly one of the dimensions, denoted by Δ_1 in Eq. (6.24), is negative, and the others are positive. Existence of a negative dimension implies that the fourth-order structure function in models (1.1)–(1.3) exhibits the inertial-range anomalous scaling; for small mr it has the form

$$S_4(\mathbf{r}) = D_0^{-2} r^{4-2\varepsilon} \sum_{\alpha} A_{\alpha} (mr)^{\Delta_{\alpha}} + \dots, \quad (6.26)$$

see Eq. (5.2). The dots stand for the corrections of the form $(mr)^{2+O(\varepsilon)}$ and higher, which arise from the operators including more derivatives than fields, and possible anisotropic contributions, related to nonscalar operators. The leading term, singular for $mr \rightarrow 0$, is determined by the negative dimension Δ_1 .

The dimensions diverge for $d \rightarrow \sqrt{3}$ as a result of the divergence of g_* in Eq. (4.9). For $d \rightarrow \infty$, they simplify and form three groups that tend to 0, ε and 2ε . More precisely, in the first group there are two dimensions with the behavior $\Delta = \pm 2\sqrt{2}\varepsilon/d + O(1/d^2)$. This means that for $d = \infty$, the anomalous scaling of S_4 vanishes: the phenomenon known for the scalar Kraichnan model [42] and questioned for the NS turbulence [43,44]. In the following section we shall see that the simplification of the exponents and vanishing of the anomalous scaling for $d \rightarrow \infty$ also holds for the higher structure functions.

Although the above expressions for the eigenvalues are well defined for any $d > \sqrt{3}$, low integer dimensions require special care because of additional linear relations between the operators. For $d=3$, there are two such relations: $\Phi_5 - \Phi_1/2 - \Phi_2 + 2\Phi_4 = 0$ and $\Phi_6 - \Phi_3/2 = 0$. For $d=2$, one more relation arises: $\Phi_4 - \Phi_1/2 = 0$. [It is also noteworthy that the derivative on the right-hand side of Eq. (6.8) in two and three dimensions vanishes identically.]

Using these relations, one can check that two basis operators (6.2) for $d=3$ and three basis operators for $d=2$ vanish. Therefore, the corresponding eigenvalues are in fact meaningless and should be discarded [in particular, this happens with the dimension (6.25)]. The eigenvalues that survive for $d=2$ and 3 belong to the three and four *lowest* branches in Fig. 3, respectively (they are denoted by the thick dots). In particular, the dangerous operator remains nontrivial, so that our model exhibits the anomalous scaling also in two and three dimensions. We recall that similar behavior was demonstrated by the zero-mode solutions: all d -dimensional expressions for the exponents have well-defined limits for $d \rightarrow 2$, but a part of them becomes in fact spurious owing to the vanishing of the corresponding amplitudes; see the discussion in the end of Sec. III B.

C. Scalar operators of the form $(\partial\theta)^{2n}$ and the anomalous scaling of S_{2n}

The leading terms of the inertial-range behavior of a higher-order structure function S_{2n} are related to the scalar composite operators of the form $(\partial\theta)^{2k}$ with $0 < k \leq n$. In the previous section we have established the anomalous scaling behavior of the fourth-order structure function, as a result of the existence of dangerous operator with $k=2$ in the corresponding OPE. Then the probabilistic inequalities allow one to show that all the higher-order structure functions are also anomalous, the leading term of the inertial-range behavior for the function S_{2n} is given by an operator with $k=n$, the number of dangerous operators is necessarily infinite, and the spectrum of their dimensions is not restricted from below.

Let Δ_n be the dimension of the operator that gives the leading contribution to the OPE for the function S_{2n} , so that $S_{2n} \propto D_0^{-n} r^{n(2-\varepsilon)} (mr)^{\Delta_n}$. It is well known in the probability theory that $|\langle x^n \rangle|^{1/n}$ is a nondecreasing function of n for any random variable x . Taking $x = [\theta_r(t, \mathbf{x}) - \theta_r(t, \mathbf{x}')]^2$ we find that $S_{2n}^{1/n} \propto D_0^{-1} r^{(2-\varepsilon)} (mr)^{\Delta_n/n}$ is a nondecreasing function of n , and so is the ratio $|\Delta_n|/n$ [we recall that Δ_n is negative and (mr) is small]. This proves all the above statements.

In principle, calculation of the critical dimensions related to the family $(\partial\theta)^{2n}$ for any given n is a purely technical

problem, and the formulas (6.11)–(6.16) remain valid in the general case with obvious alterations. In practice, however, this problem appears very cumbersome, in particular, because the number of relevant operators increases rapidly with n . The situation simplifies for large d , and below we restrict ourselves with the zeroth and first terms of the $1/d$ expansion. To avoid possible misunderstandings, it should be stressed that we deal with the $1/d$ expansion of a critical dimension Δ in its $O(\varepsilon)$ approximation, that is, the $1/d$ expansion of the coefficient $\Delta^{(1)}(d)$ in the representation $\Delta = \varepsilon\Delta^{(1)}(d) + O(\varepsilon^2)$.

Despite this simplification, no explicit analytical result is available for general n . Below we only present the results for the critical dimensions of the families $(\partial\theta)^{2n}$ with $n \leq 6$; the detailed derivation is given in Appendix B.

The first two terms of the $1/d$ expansion for such operators have the form $\Delta^{(1)}(d) = 2k + \Delta^{(11)}/d + O(1/d^2)$, where $k=0,1,\dots,n$ and $\Delta^{(11)}$ are numerical coefficients independent of the parameters ε and d . It is clear that for large d , dangerous operators can only be present in the subset with $k=0$. As already mentioned in Sec. VI B, in the family with $n=2$ there are two such operators with

$$\Delta^{(11)} = \pm 2\sqrt{2}. \quad (6.27a)$$

In the family with $n=3$ there are three such operators with

$$\Delta^{(11)} = -9.674, -0.973, 7.647. \quad (6.27b)$$

In the family with $n=4$ there are five such operators with

$$\Delta^{(11)} = -20.617, -7.783, -1.018, 3.883, 14.534. \quad (6.27c)$$

In the family with $n=5$ there are seven such operators with

$$\Delta^{(11)} = -35.589, -18.660, -8.700, \\ -2.960, 2.780, 10.674, 23.455. \quad (6.27d)$$

In the family with $n=6$ there are eleven such operators with

$$\Delta^{(11)} = -54.572, -33.612, -19.554, -13.834, -12.815, \\ -4.908, 3.839, 4.828, 9.681, 19.552, 34.395. \quad (6.27e)$$

These results confirm and illustrate the general picture outlined above: in the set of operators with $k \leq n$, the most dangerous operator (that is, the operator with the lowest negative dimension) belongs to the subset with $k=n$, and its dimension $\Delta_n < 0$ decreases faster than linearly with n . The results (6.27) are illustrated by Fig. 4 (only the negative dimensions are shown). It suggests that the dimensions form a set of monotonous branches, denoted by dashed lines. The solid curve corresponds to the well-known expression for the scalar Kraichnan model [4,5] in the same $O(\varepsilon/d)$ approximation: $\Delta_n = -2n(n-1)\varepsilon/d$. For all n , it lies below the lowest vector branch (the scaling in the scalar model appears “more anomalous”); the deviation between the scalar and vector cases becomes stronger as n increases, although the *ratio* of the dimensions approaches unity.

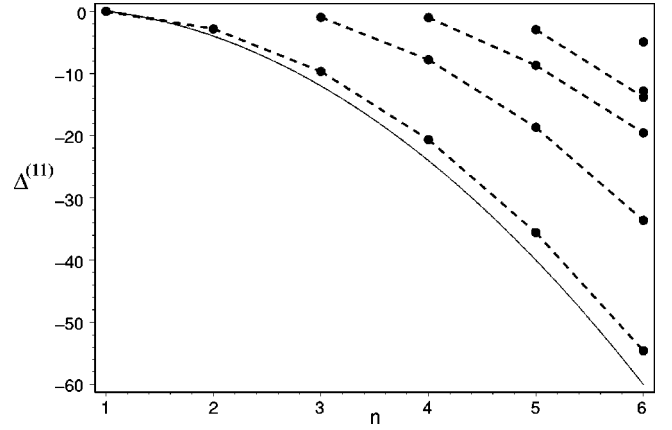


FIG. 4. Coefficients $\Delta^{(11)}$ in the $O(\varepsilon/d)$ approximation of the critical dimensions of the operators $(\partial\theta)^{2n}$ in the scalar (solid curve) and vector (thick dots) models. Dashed lines denote monotonous branches of the critical dimensions in the vector case.

D. Tensor operators and the scaling of S_2 in anisotropic sectors of arbitrarily high orders

In this section, we apply the RG and OPE approach to the higher anisotropic sectors of the model (1.1)–(1.3). We shall concentrate on the second-order structure function, for which nonperturbative results can, in principle, be derived for arbitrarily high values of the parameter l from the exact Dyson-Wyld equation (see Sec. III B). This allows one to identify the solutions of the zero-mode equations, discussed in Sec. III, with definite composite operators and, in principle, to calculate the corresponding amplitude factors. Using the OPE technique, we derive explicit analytical expressions for the leading exponents in all anisotropic sectors to the order $O(\varepsilon)$ in d dimensions. Furthermore, we present additional nontrivial exponents that do not appear in the inertial-range behavior of the model (1.1)–(1.3), but will be activated (and can determine *leading* terms in anisotropic sectors) if the anisotropy is introduced by the velocity field (like in Ref. [13]) and not only by the large-scale forcing.

The analysis of the anisotropic sectors for higher-order functions using the OPE is extremely cumbersome but, in a sense, purely technical problem; we shall briefly discuss it in Sec. VI E.

According to the general rules (see Sec. V), the leading terms in the sector $l=2$ are determined by the second-rank operators built of two gradients; up to derivatives, there are two such operators:

$$F_1 = \mathcal{P}_{\text{irr}}[\partial_i \theta_k \partial_j \theta_k], \quad F_2 = \mathcal{P}_{\text{irr}}[\partial_k \theta_i \partial_k \theta_j], \quad (6.28)$$

where \mathcal{P}_{irr} denotes the irreducible part; cf. Eq. (3.11) in Sec. III B (here and below, we use F_α and Δ_α to denote different operators and their dimensions).

We omit the one-loop calculation of the corresponding 2×2 matrix of critical dimensions, which is similar to the calculation discussed in Sec. VI B for scalar operators, and present only its eigenvalues, i.e., the critical dimensions of operators (6.28),

$$\Delta_{1,2} = \varepsilon \frac{d^3 + 5d^2 + 2d - 8 \pm \sqrt{(d+4)(d^5 + 2d^4 - 7d^3 - 4d^2 + 8d + 16)}}{2(d^2 - 3)(d+4)} + O(\varepsilon^2). \quad (6.29)$$

In representation (5.2), they give rise to exponents $2 - \varepsilon + \Delta_{1,2}$, which agree with the special case $l=2$ of expressions (3.22) and (3.23), obtained in Sec. III B on the basis of the Dyson-Wyld equation.

In two dimensions, the transverse vector field can be represented in the form

$$\theta_i = \epsilon_{ik} \partial_k \psi, \quad (6.30)$$

where ϵ_{ik} is the antisymmetric Levi-Civita pseudotensor and $\psi(x)$ is some scalar function (stream function). Using the well known identity $\epsilon_{ik} \epsilon_{js} = \delta_{ij} \delta_{ks} - \delta_{is} \delta_{kj}$ one can easily check that for $d=2$, the operators (6.28) coincide. Only one of the dimensions (6.29), namely, $\Delta_2 = \varepsilon$, corresponds to a nontrivial basis operator (6.2) and remains meaningful, while the other, $\Delta_1 = 3\varepsilon$, corresponds to a vanishing basis operator and should be discarded; cf. Fig. 2 for $d=l=2$.

The leading terms in the sector $l=4$ are determined by the fourth rank operators, obtained from the monomial $F_{ijkn} \equiv \mathcal{P}_{\text{int}}[\partial_i \theta_j \partial_k \theta_n]$ by all possible permutations of its tensor indices. It turns out that there are only *three* different critical dimensions, associated with these operators. The corresponding basis operators (6.2) possess different symmetries and therefore can be written down without calculation of diagrams. One of them is the fully symmetrized operator,

$$F_S = F_{ijkn} + F_{jink} + F_{ikjn} + F_{kinj} + F_{jikn} + F_{iknj}.$$

The others can be constructed as follows. The monomials can be split into three groups of two operators each,

$$F_1 = F_{ikjn}, \quad F_2 = F_{kinj},$$

symmetric with respect to the simultaneous exchange of the indices within the pairs $\{ij\}$ and $\{kn\}$,

$$F_3 = F_{ijkn}, \quad F_4 = F_{jink},$$

symmetric with respect to the exchange of the indices within the pairs $\{ik\}$ and $\{jn\}$,

$$F_5 = F_{jikn}, \quad F_6 = F_{iknj},$$

symmetric with respect to the exchange within the pairs $\{in\}$ and $\{jk\}$.

The operators can mix in renormalization only within the groups with the same symmetry, so that the set F_1, F_2, F_S is closed with respect to the renormalization, and so are the sets F_3, F_4, F_S and F_5, F_6, F_S . In the first set, the basis operators (6.2) are F_S (it is fully symmetric and no other operators can admix to it), $F_1 - F_2$ (it is antisymmetric with respect to the exchange of the indices within the pairs $\{ik\}$ and $\{nj\}$), and $F_S - 3F_1 - 3F_2$. The latter operator is symmetric with respect to the exchange of the indices within the pairs $\{ik\}$ and $\{nj\}$, and for this reason it cannot mix with

the second basis operator; it is not fully symmetric and cannot admix to F_S . It remains to note that the contraction of the operators $F_1 - F_2$ and $F_S - 3F_1 - 3F_2$ with any constant vector with respect to all four indices gives zero, while analogous contraction of F_S remains nontrivial. This means that F_S cannot admix to those operators in renormalization, and the above construction indeed gives three independent basis elements of the type (6.2).

The explicit one-loop calculation confirms this conclusion and gives three different critical dimensions, corresponding to the basis operators F_S , $F_S - 3F_1 - 3F_2$, and $F_1 - F_2$, respectively:

$$\Delta_1 = \frac{(d+2)(d^2 + 4d - 9)}{(d^2 - 3)(d+6)} \varepsilon, \quad (6.31a)$$

$$\Delta_2 = \frac{(d^2 - 1)}{(d^2 - 3)} \varepsilon, \quad (6.31b)$$

$$\Delta_3 = \frac{(d^3 + 4d^2 - d - 8)}{(d^2 - 3)(d+4)} \varepsilon, \quad (6.31c)$$

with corrections of order $O(\varepsilon^2)$ and higher.

The remaining independent basis operators can be obtained by permutations of the indices and can be chosen in the form $F_3 - F_4$, $F_5 - F_6$, and $F_S - 3F_3 - 3F_4$. At first glance, it seems that one can add another independent element, $F_S - 3F_5 - 3F_6$, but in fact it is equal to the sum of the operators $F_S - 3F_1 - 3F_2$ and $F_S - 3F_3 - 3F_4$ up to the minus sign. Therefore the dimension Δ_1 in Eq. (6.31) is unique, Δ_2 has two fold degeneracy and Δ_3 has threefold degeneracy.

Although all three dimensions in Eq. (6.31) make sense, only Δ_1 appears on the right-hand side of expansion (5.2). Indeed, both the diagrammatic analysis and dimensional considerations show that the coefficients in Eq. (5.1), corresponding to operators F_{ijkn} , do not involve the function $\langle \theta \theta \rangle_0$ from Eq. (2.4b). Therefore, they do not depend on the vector \mathbf{n} and their tensor indices are carried by the Kronecker δ symbols or vectors \mathbf{r} . The contraction of any *irreducible* operator with the δ symbols always gives zero; the contraction with the components of a single vector \mathbf{r} in expansion (5.1) ‘‘kills’’ the basis operators of the form $F_S - 3F_1 - 3F_2$ and $F_1 - F_2$ (see above).

Thus the only contribution to expansion (5.2) comes from the operator F_S , and the expression $2 - \varepsilon + \Delta_1$ should be identified with the leading exponent γ_4 from Eq. (3.21) in Sec. III; they indeed agree in the order $O(\varepsilon)$.

The remaining dimensions $\Delta_{2,3}$ will be activated [and can determine *leading* terms on the right-hand side of Eq. (5.2) in the $l=4$ sector] if the vector \mathbf{n} appears in the correspond-

ing coefficients C_F in expansion (5.1). This can happen if the anisotropy is introduced by the velocity field (like in Ref. [13]) and not only by the large-scale forcing. However, it should be noted that in such a case the dimensions become nonuniversal through their dependence on the anisotropy parameters, and the expressions like Eq. (6.31) give only the zeroth order (isotropic) approximations; cf. Ref. [13] for the scalar case.

Now let us turn to general l . The relevant operators are the l th rank tensors built of two fields θ and minimal possible number of derivatives:

$$\mathcal{P}_{\text{int}}[\theta_{i_1}\partial_{i_2}\cdots\partial_{i_{l-1}}\theta_{i_l}]. \quad (6.32a)$$

Up to total derivatives, monomials (6.32a) are invariant with respect to the shift $\theta \rightarrow \theta + \text{const}$, so that their dimensions can appear in expansion (5.2). In the above form, all symmetries of operators (6.32a) are obvious: they are symmetric with respect to the permutation of the indices $\{i_1, i_l\}$ and any permutations within the subset $\{i_2 \cdots i_{l-1}\}$. Thus the total number of *different* monomials, obtained from Eq. (6.32a) by permutations of the indices, equals $l(l-1)/2$. However, there are only three different dimensions related to them. Indeed, the counterterm to the monomial (6.32a) necessarily has the same symmetries and therefore can include, along with Eq. (6.32a) itself, two more structures:

$$\begin{aligned} & \text{Sym } \mathcal{P}_{\text{int}}[\theta_{i_2}\partial_{i_1}\partial_{i_3}\cdots\partial_{i_{l-1}}\theta_{i_l}], \\ & \text{Sym } \mathcal{P}_{\text{int}}[\theta_{i_2}\partial_{i_1}\partial_{i_2}\partial_{i_4}\cdots\partial_{i_{l-1}}\theta_{i_3}], \end{aligned} \quad (6.32b)$$

where Sym denotes the symmetrization with respect to the permutation of the pair $\{i_1, i_l\}$ and any permutations of the indices $\{i_2 \cdots i_{l-1}\}$. Thus the set of three operators (6.32) is closed with respect to the renormalization, the corresponding basis operators (6.2) are their linear combinations and determine three different dimensions $\Delta_{1,2,3}$; all the other basis operators are obtained by permutations of the indices and give rise only to the same dimensions.

For general l , even the one-loop calculation is rather difficult because individual contributions in the counterterms to the polynomials (6.32) contain powerlike UV divergencies in addition to logarithmic ones. In contrast with the calculation discussed in Sec. VI B, one cannot neglect the space-time inhomogeneity of the fields θ in the diagram; in other words, one cannot neglect the dependence of its integrand on the external momenta \mathbf{p} and should expand the integrand up to the terms of order p^{2l-2} . This expansion gives rise to the terms that diverge for $\Lambda \rightarrow \infty$ as some positive powers of the UV cutoff Λ . However, all such terms contain δ symbols and cancel out when all contributions in irreducible structures (6.32) are taken into account; the result is finite at $\Lambda \rightarrow \infty$, contains a first-order pole in ε , and reduces to a linear combination of the three structures (6.32a) and (6.32b).

We omit the details of this cumbersome calculation and give only the result,

$$\begin{aligned} \Delta_\alpha &= (l-4) + \frac{\varepsilon x_\alpha}{(d^2-3)(d+2l-6)(d+2l-4)(d+2l-2)} \\ &+ O(\varepsilon^2), \end{aligned} \quad (6.33)$$

where

$$\begin{aligned} x_1 &= d^5 + 6d^4l + 13d^3l^2 + 10d^2l^3 + dl^4 - 12d^4 - 53d^3l \\ &\quad - 60d^2l^2 - 6dl^3 + 2l^4 + 47d^3 + 92d^2l - 29dl^2 - 44l^3 \\ &\quad - 24d^2 + 158dl + 214l^2 - 156d - 364l + 192, \\ x_2 &= (d+2l-4)(d+2l-2)(d^3 + 2d^2l + dl^2 - 6d^2 - 5dl \\ &\quad + 2l^2 + 3d - 16l + 30), \\ x_3 &= (d+2l-2)(d^4 + 4d^3l + 5d^2l^2 + dl^3 - 10d^3 - 25d^2l \\ &\quad - 5dl^2 + 2l^3 + 27d^2 - 6dl - 26l^2 + 30d + 92l - 96). \end{aligned}$$

For $l=4$, the results (6.31) are recovered. Like in the case $l=4$, the basis operator (6.2) that possesses the dimension Δ_1 is symmetric with respect to all possible permutations of the full set $\{i_1 \cdots i_l\}$; only this operator survives the contraction with the coefficients C_F in the operator product expansion (5.1) for the model (1.1)–(1.3). Therefore only Δ_1 appears on the right-hand side of Eq. (5.1) and determines the leading exponent for the l th anisotropic sector: $\gamma_l = 2 - \varepsilon + \Delta_1$. For all l and d , this recovers the result (3.21) obtained in Sec. III on the basis of the Dyson-Wyld equations.

Like in the case $l=4$, the remaining dimensions $\Delta_{2,3}$ in Eq. (6.33) are activated and appear on the right-hand side of Eq. (5.2) when the anisotropy is introduced by the velocity field.

We recall that the general form of the exponent for a given $l \geq 2$ is $\gamma_l = l - 2 + 2k + O(\varepsilon)$; see Sec. III B. From the OPE viewpoints, $k=0$ corresponds to the l th rank irreducible operator built of two fields θ and $l-2$ derivatives, symmetric in all indices (see above). The operators with $k=1, 2, \dots$ can be obtained in two ways: one can add k Laplacians to the operator described above, or one can add $(k-1)$ Laplacians, two derivatives with free indices, and contract the indices of the θ fields. Therefore, for general d the leading exponent $\gamma_l = l - 2 + O(\varepsilon)$ is unique, while for all $k=1, 2, \dots$ there are *two* correction exponents of the form $\gamma_l = l - 2 + 2k + O(\varepsilon)$. In two dimensions, these two possibilities coincide, see the discussion below Eq. (6.29), and only one exponent exists for any $k=0, 1, 2, \dots$.

For $l=0$, the general form of the exponent is $\gamma = 2k + O(\varepsilon)$; the first two solutions, $\gamma=0$ and $\gamma=2-\varepsilon$, are known exactly both from the RG and the Dyson-Wyld equation, see Sec. III A and VI A. For any k and d , the solution is unique: for the scalar operators, the indices of the fields are contracted, and the operator with $\gamma = 2k + O(\varepsilon)$ necessarily reduces to the unique form $\theta_i \Delta^k \theta_i$.

This picture is in a full agreement with the results obtained in Sec. III for general d and l on the basis of the exact Dyson-Wyld equation for the pair correlation function; see also Ref. [26] for $d=3$ and $j \leq 10$.

E. Higher-order structure functions in the higher-order anisotropic sectors

Let us briefly discuss the scaling behavior in the anisotropic sectors for the higher-order structure functions. The RG and OPE analysis given in Sec. V and VI D can directly be extended to the general case. It shows that the leading exponents γ_{nl} in the l th sector of the $2n$ -order structure function, $S_{2n} \propto P_l(z) r^{\gamma_{nl}}$, are determined by the l th rank tensor operators with $k \leq 2n$ fields θ and minimal possible number of derivatives; the operators that contain the field θ without derivative, or reduce to total derivatives, give no contribution to the expansion (5.2) and should be discarded.

The practical calculation of the critical dimensions of the operators with large l or n is a difficult task, as one could see already on the example with $l=0$ and $n=2$. However, in the zeroth order of the ε expansion, some important information can be obtained without calculation, just by the analysis of the form of relevant operators.

One can easily see that for $l \leq 4n$, the leading exponents are determined by the l th rank tensor operators of the form $(\partial\theta)^{2n}$, with l free and $4n-l$ contracted indices. In the $O(1)$ approximation, the exponents themselves are equal to the number of derivatives entering into the operators: $\gamma_{nl} = 2n + O(\varepsilon)$. For $l > 4n$, the relevant operators necessarily contain more derivatives than fields, and for the leading exponents one obtains: $\gamma_{nl} = l - 2n + O(\varepsilon)$.

One can thus conclude that for higher n , the general picture remains the same as for S_2 : each anisotropic sector possesses its own set of scaling exponents; the leading exponents obey hierarchy relations at least for small ε and $l > 4n$; they grow with l without bound. Of course, there can be several exponents for given l and n ; we recall that there are six exponents of the form $\gamma_{20} = 4 + O(\varepsilon)$ for $l=0$ and $n=2$ (see Sec. VI C). In order to identify the unique leading exponent within a family with the same zeroth-order value, or to verify the hierarchy relations for $l \leq 4n$, one should perform the $O(\varepsilon)$ calculation for the relevant families of operators. This cumbersome task lies beyond the scope of the present paper and will be discussed elsewhere.

VII. CONCLUSION

We have studied the inertial-range scaling behavior in a model of the passive vector quantity advected by a self-similar white-in-time Gaussian velocity field, with the large-scale anisotropy introduced by a random forcing. In two respects, the model is closer to the real Navier-Stokes turbulence than the famous scalar rapid-change model: nonlocality of the dynamics and mixing of the composite operators that determine anomalous exponents.

The incompressibility condition for the advected field and the pressure term in the diffusion-convection equation make the dynamics nonlocal. This raises the question of realizability of the zero-mode solutions, that is, convergence of the integrals in the equations for the correlation functions on powerlike solutions, and consistency of nonlocality and the existence of infinite families of scaling exponents [26,36]. The detailed analysis of the exact integral equation satisfied by the pair correlation function has shown that the general

picture of the inertial-range scaling is essentially the same as in the scalar [14] and magnetic [23,29,30] variants of the rapid-change model. Namely, each anisotropic sector is described by an infinite set of scaling exponents, with the spectrum unbounded from above. The leading exponents in each sector are organized in the hierarchical order according to their degree of anisotropy, with the main contribution coming from the isotropic sector in agreement with the hypothesis on the restored local isotropy of the fully developed turbulence [1]. The leading exponents themselves grow without bound with the degree of anisotropy, in disagreement with the idea of the window of locality [36].

The integral operator entering into the equation for the pair correlation function in the momentum space converges on the powerlike solutions with the *leading* exponents in the $l=0$ and $l=2$ sectors, but formally diverges on powerlike solutions with *subleading* exponents and *leading* exponents in the sectors with $l \geq 4$. However, correct analysis of convergence here requires the knowledge of the behavior of the full solution beyond the inertial range, where it no longer reduces to a sum of power terms. It turns out, that natural assumptions about the form of the solution allow one to perform certain subtractions in the integrals that make them convergent. One can use the formal rules of analytical regularization and simultaneously omit the subtracted terms to obtain correct answers for the convergent integrals with proper subtractions. Moreover, the realizability of these solutions is also guaranteed by the RG approach, where they are identified with the contributions of certain composite operators in the corresponding operator product expansions. Therefore, such exponents indeed appear in the full solution in the inertial range.

These conclusions are in agreement with the recent analysis performed in Ref. [26] for the model (1.1)–(1.3) in the three-dimensional coordinate space, although the analysis in the momentum space appears rather different; see also Ref. [24] for the general vector model. Furthermore, the RG and OPE techniques confirm this picture and extend it to the higher-order correlation functions.

The second aim of the paper has been the analysis of the anomalous scaling of the higher-order even structure functions S_{2n} . Owing to the conservation of the ‘‘energy’’ $\theta^2(x)$, the second-order function appears nonanomalous with the simple dimensional exponent: $S_2 \propto r^{2-\varepsilon}$. The anomalous scaling reveals itself on the level of the fourth-order structure function. In contrast with the scalar case, where the leading anomalous exponents were identified with the critical dimensions of individual composite operators in the corresponding OPE [8], the vector nature of the advected field in our model leads to mixing of operators. In particular, the inertial-range behavior of the function S_4 in d dimensions is given by a set of six close exponents, determined by eigenvalues of the matrix of critical dimensions for a set of six operators. One of the dimensions is negative (‘‘dangerous operator’’) and gives rise to anomalous scaling. The number of relevant operators increases rapidly with the order of the function; they have been calculated in a controlled approximation (small ε and large d) for the higher-order functions up to S_{12} . The latter involves as many as 16 negative exponents, ten of them

coming from the lower-order functions, and a multitude of positive exponents that are small and therefore close to the negative ones for small ε . The probabilistic inequalities prove that all the higher-order structure functions are also anomalous, and the total number of dangerous operators in our model is infinite, with the spectrum of dimensions unbounded from below.

Since the mixing of operators is a manifestation of the vector nature of the advected field, there are a few general conclusions regarding the real NS turbulence, which one can draw from the analysis of model (1.1)–(1.3).

It was demonstrated recently that a careful disentangling of contributions from different anisotropic sectors is unavoidable in the analysis of experimental data on the real turbulence, because it allows one to properly identify scaling exponents in the situations, where the standard treatment reveals no scaling behavior at all; see the discussion in Refs. [32–35] and references therein. The example of the model (1.1)–(1.3) shows that even in the isotropic sector, or for the ideal isotropic turbulence, correlation functions are represented by infinite sums of powerlike terms, and the number of close terms grows rapidly with the order of the correlation function. Although these corrections die out in the formal limit $L \rightarrow \infty$, and a pure powerlike behavior with the leading exponent sets in, in practice it may be obscured by such corrections: the subleading exponents can be very close to the leading ones and much more important than the leading terms from the higher anisotropic sectors. This might result in imaginary nonuniversality of the inertial-range exponents or deviations from a pure scaling behavior, which increase with the order of the correlation function. Therefore, reliable analysis of the inertial-range scaling necessarily requires some theory for the correction exponents.

In theoretical models, anomalous scaling is usually explained by the so-called intermittency phenomenon. Within the framework of numerous models, the anomalous exponents are related to the statistics of the local dissipation rate or to the dimensionality of fractal structures formed by small-scale vortices in the dissipative range; the detailed review and bibliography can be found in Ref. [1]. As a rule, those theories predict simple analytic formulas for the dependence of the anomalous exponents on n , the order of the structure function. Although such formulas can provide a very good fit for the experimental results, the experience on the rapid-change models suggests that they cannot be absolutely correct. Even for the scalar model, the n dependence of the anomalous exponents changes as the order of the ε increases [8–12]. Moreover, for our vector problem the n dependence of the exponents can hardly be given by a single explicit formula (except probably for the case $d=2$, where the model can be mapped onto a nonlocal scalar problem): the relevant families of operators are completely different for different n , so that each function S_n requires special analysis.

Recently, a systematic perturbation theory for the anomalous exponents in the NS turbulence was proposed, where the role of a formal expansion parameter is played by the anomalous exponent for S_2 , assumed to be small but nontrivial [45]. In contrast with the magnetic variant of the rapid-change model [27,28] and the general “A model” with

a stretching term [24], the second-order structure function in our vector problem (1.1)–(1.3) is nonanomalous, and the perturbation theory of Ref. [45] would be impossible here. Like in the scalar model [3–5,7], this is a consequence of the energy conservation or, in the field-theoretic language, of vanishing of the critical dimension of the local dissipation rate; see Sec. VI A for the vector and Ref. [8] for the scalar cases. The vanishing of the critical dimension of the dissipation rate at the physical value of ε is also characteristic of the NS case [46,47], which raises serious doubts about the existence of the second-order anomaly and the possibility of the corresponding perturbation theory in the NS turbulence.

The analysis of the inertial-range behavior essentially simplifies as $d \rightarrow \infty$. Our model has no finite “upper critical dimension,” above which anomalous scaling vanishes (see Ref. [48] for a recent discussion of that concept). Like in the scalar case [42] and, probably, in the NS turbulence [43,44], the anomalous scaling disappears at $d = \infty$, but it reveals itself already in the $O(1/d)$ approximation. The anomalous exponents can be calculated within the double expansion in ε and $1/d$. Along with the results [4] for the scalar rapid-change model, this confirms the importance of the large- d expansion for the issue of anomalous scaling in fully developed turbulence.

Although our analysis has been confined to the linear problem (1.1)–(1.3), which has only restricted resemblance with the real fluid turbulence, some of the results can be extended to the case of the vector passive field advected by the NS field, or the nonlinear NS equation itself with some classes of random forcing. These questions lie beyond the scope of the present paper. Detailed exposition of the RG approach to the NS problem and the bibliography can be found in Refs. [16,17]; the renormalization of composite operators and the concept of the operator product expansion are also discussed in Refs. [15,41,44,46,47,49,50]. In particular, critical dimensions of tensor composite operators in the stirred NS problem were calculated in Ref. [50] (see also Sec. 2.3 of [17]); they demonstrate the same hierarchy as their counterparts from Sec. VI D.

We believe that the framework of the renormalization group and operator product expansion, the concept of dangerous composite operators, exact functional equations, and the ε and $1/d$ expansions will become the necessary elements of the appearing theory of the anomalous scaling in fully developed turbulence.

ACKNOWLEDGMENTS

The authors are thankful to M. Hnatich, J. Honkonen, A. Kupiainen, A. Mazzino, P. Muratore Ginanneschi, S. V. Novikov, and A. N. Vasil’ev for discussions. The work was supported by the Russian Foundation for Fundamental Research (Grant No. 99-02-16783), the Grant Center for Natural Sciences of the Russian State Committee for Higher Education (Grant No. E00-3-24), and the Nordic Grant No. FIN-18/2001.

APPENDIX A: EQUATIONS FOR THE EXPONENTS IN THE HIGHER ANISOTROPIC SECTORS OF THE PAIR CORRELATION FUNCTION

It was shown in Sec. III B that the equation for the exponents γ_l in l th anisotropic sector can be written as

$\det|C_{\alpha\beta}^{(l)}|=0$. For $l=2$, the matrix elements $C_{\alpha\beta}^{(l)}$ were given in Eq. (3.19). In general, the matrix $C_{\alpha\beta}^{(l)}$ is symmetric and its elements are finite linear combinations of the integrals $\tilde{I}_1 \equiv I_1 + 2J$ with J from Eq. (2.13) and $I_n \equiv I_n(\gamma_l - 2, \varepsilon + 2)$ from Eq. (3.3), with n as high as $l/2 + 2$. In this appendix, we present the coefficients $C_{\alpha\beta}^{(l)}$ for higher values of l up to $l=12$. Then equations for γ_l can be written in a straightforward way. Below we denote $d_{k,s} = (d+k)(d+k+2)\cdots(d+s-2)(d+s)$ and $d_k = (d+k)$.

$$C_{11}^{(4)} = d_{2,4}I_4 - (d+2)(d^2+5d-2)I_3 + (d^2-1)(2d+5)I_2 - (d-1)^2(d+1)\tilde{I}_1,$$

$$C_{12}^{(4)} = -d_{2,4}I_4 + 2d_{2,4}I_3 - (d+1)(2d+5)I_2 + (d^2-1)\tilde{I}_1,$$

$$C_{22}^{(4)} = d_{2,4}I_4 - \frac{1}{2}d_{2,4}(d+2)I_3 + \frac{(d+1)}{12}(d^3+10d^2 + 24d+12)I_2 - \frac{d}{12}(d^2-1)(d+4)\tilde{I}_1,$$

$$C_{11}^{(6)} = -I_5 + \frac{(d^2+10d+1)}{d_8}I_4 - 3\frac{d_3(d^2+6d-5)}{d_{6,8}}I_3 + \frac{d_{1,3}(d-1)(3d+13)}{d_{4,8}}I_2 - \frac{d_{1,3}(d-1)^2}{d_{4,8}}\tilde{I}_1,$$

$$C_{12}^{(6)} = I_5 - \frac{(3d+19)}{d_8}I_4 + \frac{d_3(4d+25)}{d_{6,8}}I_3 - \frac{d_{1,3}(3d+13)}{d_{4,8}}I_2 + \frac{d_{1,3}(d-1)}{d_{4,8}}\tilde{I}_1,$$

$$C_{22}^{(6)} = -I_5 + \frac{(d^2+14d+43)}{3d_8}I_4 + \frac{d_3(d^3+30d^2+216d+430)}{30d_{6,8}}I_3 + \frac{d_{1,3}(d^3+15d^2+66d+85)}{15d_{4,8}}I_2 - \frac{d_{1,3}(d-1)(d^2+8d+10)}{30d_{4,8}}\tilde{I}_1,$$

$$C_{11}^{(8)} = I_6 - \frac{(d^2+15d+8)}{d_{12}}I_5 + 2\frac{d_5(2d-1)(d+11)}{d_{10,12}}I_4 - 2\frac{d_{3,5}(3d^2+23d-22)}{d_{8,12}}I_3 + \frac{d_{1,5}(d-1)(4d+25)}{d_{6,12}}I_2 - \frac{d_{1,5}(d-1)^2}{d_{6,12}}\tilde{I}_1,$$

$$C_{12}^{(8)} = -I_6 + 2\frac{(2d+17)}{d_{12}}I_5 - \frac{d_5(7d+64)}{d_{10,12}}I_4 + \frac{d_{3,5}(7d+58)}{d_{8,12}}I_3 - \frac{d_{1,5}(4d+25)}{d_{6,12}}I_2 + \frac{d_{1,5}(d-1)}{d_{6,12}}\tilde{I}_1,$$

$$C_{22}^{(8)} = I_6 - \frac{(d^2+24d+116)}{4d_{12}}I_5 + \frac{d_5(d^3+58d^2+748d+2632)}{56d_{10,12}}I_4 - \frac{d_{3,5}(3d^3+90d^2+788d+2072)}{56d_{8,12}}I_3 + \frac{d_{1,5}(3d^2+50d+196)}{56d_{6,12}}I_2 - \frac{d_{1,5}(d-1)(d^2+12d+28)}{56d_{6,12}}\tilde{I}_1,$$

$$C_{11}^{(10)} = -I_7 + \frac{(d+1)(d+19)}{d_{16}}I_6 - 5\frac{d_7(d^2+15d-4)}{d_{14,16}}I_5 + 10\frac{d_{5,7}(d^2+12d-9)}{d_{12,16}}I_4 - 5\frac{d_{3,7}(2d^2+19d-19)}{d_{10,16}}I_3 + \frac{d_{1,7}(d-1)(5d+41)}{d_{8,16}}I_2 - \frac{d_{1,7}(d-1)^2}{d_{8,16}}\tilde{I}_1,$$

$$C_{12}^{(10)} = I_7 - \frac{(5d+53)}{d_{16}}I_6 + \frac{d_7(11d+128)}{d_{14,16}}I_5 - 2\frac{d_{5,7}(7d+81)}{d_{12,16}}I_4 + \frac{d_{3,7}(11d+113)}{d_{10,16}}I_3 - \frac{d_{1,7}(5d+41)}{d_{8,16}}I_2 + \frac{d_{1,7}(d-1)}{d_{8,16}}\tilde{I}_1,$$

$$C_{22}^{(10)} = -I_7 + \frac{(d^2+36d+239)}{5d_{16}}I_6 - \frac{d_7(d^3+94d^2+1842d+9432)}{90d_{14,16}}I_5 + 2\frac{d_{5,7}(d^3+49d^2+672d+2727)}{45d_{12,16}}I_4 - \frac{d_{3,7}(d^3+34d^2+357d+1167)}{15d_{10,16}}I_3 + \frac{d_{1,7}(2d^3+53d^2+444d+1179)}{45d_{8,16}}I_2 - \frac{d_{1,7}(d-1)(d^2+16d+54)}{90d_{8,16}}\tilde{I}_1,$$

$$\begin{aligned}
C_{11}^{(12)} &= I_8 - \frac{(d^2 + 25d + 34)}{d_{20}} I_7 + 3 \frac{d_9(2d^2 + 39d - 1)}{d_{18,20}} I_6 \\
&\quad - 5 \frac{d_{7,9}(3d^2 + 49d - 28)}{d_{16,20}} I_5 + 5 \frac{d_{5,9}(4d^2 + 55d - 47)}{d_{14,20}} I_4 \\
&\quad - 3 \frac{d_{3,9}(5d^2 + 57d - 58)}{d_{12,20}} I_3 + \frac{d_{1,9}(d-1)(6d+61)}{d_{10,20}} I_2 \\
&\quad - \frac{d_{1,9}(d-1)^2}{d_{10,20}} \tilde{I}_1, \\
C_{12}^{(12)} &= -I_8 + 2 \frac{(3d+38)}{d_{20}} I_7 - \frac{d_9(16d+223)}{d_{18,20}} I_6 \\
&\quad + 5 \frac{d_{7,9}(5d+72)}{d_{16,20}} I_5 - 5 \frac{d_{5,9}(5d+69)}{d_{14,20}} I_4 \\
&\quad + 4 \frac{d_{3,9}(4d+49)}{d_{12,20}} I_3 - \frac{d_{1,9}(6d+61)}{d_{10,20}} I_2 + \frac{d_{1,9}(d-1)}{d_{10,20}} \tilde{I}_1, \\
C_{22}^{(12)} &= I_8 - \frac{(d^2 + 50d + 424)}{6d_{20}} I_7 \\
&\quad + \frac{d_9(d^3 + 138d^2 + 3768d + 25564)}{132d_{18,20}} I_6 \\
&\quad - 5 \frac{d_{7,9}(d^3 + 72d^2 + 1392d + 7744)}{132d_{16,20}} I_5 \\
&\quad + 5 \frac{d_{5,9}(d^3 + 50d^2 + 754d + 3498)}{66d_{14,20}} I_4 \\
&\quad - \frac{d_{3,9}(5d^3 + 195d^2 + 2406d + 9416)}{66d_{12,20}} I_3 \\
&\quad + \frac{d_{1,9}(5d^3 + 162d^2 + 1680d + 5588)}{132d_{10,20}} I_2 \\
&\quad - \frac{d_{1,9}(d-1)(d^2 + 20d + 88)}{132d_{10,20}} \tilde{I}_1.
\end{aligned}$$

APPENDIX B: CALCULATION OF THE CRITICAL DIMENSIONS OF THE OPERATORS $(\partial\theta)^{2n}$ FOR LARGE d

The critical dimensions of the scalar composite operators of the form $(\partial\theta)^{2n}$ were presented and discussed in Sec. VI C in the first nontrivial orders, $O(\varepsilon)$ and $O(\varepsilon/d)$, of the double expansion in ε and $1/d$; see Eqs. (6.27). Below we give the derivation of those results.

The UV divergent part of the diagram from Eq. (6.9) with a composite operator $(\partial\theta)^{2n}$ is given by the formulas (6.9)–(6.20) for $n=2$; see Sec. VI B. With obvious alterations, they apply to the case of general n and can be summarized as [the $O(g)$ approximation in the renormalized variables]:

$$\frac{gC_d(\mu/m)^\varepsilon}{2\varepsilon} \frac{\partial^2 F(a)}{\partial a_{i_1 i_2} \partial a_{i_3 i_4}} \partial_{j_1} \theta_{j_2}(x) \partial_{j_3} \theta_{j_4}(x) T_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4}, \quad (\text{B1})$$

where we have used the relations (1.4) and (6.18) and denoted

$$T_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} \equiv \int d\mathbf{n} n_{i_1} n_{i_3} P_{i_2 j_2}(\mathbf{n}) P_{i_4 j_4}(\mathbf{n}) P_{j_1 j_3}(\mathbf{n}). \quad (\text{B2})$$

The integration over the unit sphere in d -dimensional space in Eq. (B2) can be explicitly performed using the isotropy relations (6.17):

$$\begin{aligned}
T_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} &= \left[\frac{1}{d} \delta_{i_1 i_3} \delta_{i_2 j_2} \delta_{i_4 j_4} \delta_{j_1 j_3} \right. \\
&\quad - \frac{1}{d_{0,2}} (\delta_{i_2 j_2} \delta_{i_4 j_4} \delta_{i_1 i_3 j_1 j_3} \\
&\quad + \delta_{j_1 j_3} \delta_{i_2 j_2} \delta_{i_1 i_3 i_4 j_4} + \delta_{i_4 j_4} \delta_{j_1 j_3} \delta_{i_1 i_2 i_3 j_2}) \\
&\quad + \frac{1}{d_{0,4}} (\delta_{j_1 j_3} \delta_{i_1 i_2 i_3 i_4 j_2 j_4} + \delta_{i_2 j_2} \delta_{i_1 i_3 i_4 j_1 j_3 j_4} \\
&\quad \left. + \delta_{i_4 j_4} \delta_{i_1 i_2 i_3 j_1 j_2 j_3}) - \frac{1}{d_{0,6}} \delta_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} \right], \quad (\text{B3})
\end{aligned}$$

where $d_{0,k} = d(d+2) \dots (d+k)$; cf. Appendix A.

Although the coefficients of the tensor (B3) behave as $O(1/d)$ for $d \rightarrow \infty$, contractions of their indices i_k can compensate the smallness [the indices j_k are contracted with the factor $\partial\theta\partial\theta$ in Eq. (B1) and cannot be contracted with each other]. Such contractions occur when the factor $\partial^2 F(a)/\partial a \partial a$ contains the terms of the form $\delta_{i_1 i_3} \delta_{i_2 i_4}$ or $\delta_{i_1 i_4} \delta_{i_2 i_3}$ (the third term, $\delta_{i_1 i_2} \delta_{i_3 i_4}$, is forbidden by the transversality of the field θ). In particular,

$$\frac{\partial^2 \phi_1}{\partial a_{i_1 i_2} \partial a_{i_3 i_4}} = 2 \delta_{i_1 i_3} \delta_{i_2 i_4}, \quad \phi_1 \equiv \Phi_{jj}^{ii}. \quad (\text{B4})$$

Here and below, we use the notation $\Phi_{j_1 \dots j_k}^{i_1 \dots i_k} = \partial_{j_1} \theta_{i_1} \dots \partial_{j_k} \theta_{i_k}$; cf. Eq. (6.7) in Sec. VI B.

However, both such contractions can appear simultaneously only in the term with $d_{0,4}$ in Eq. (B3) and give an $O(1/d)$ contribution. The leading $O(1)$ terms arise from the contraction of the pair $i_1 i_3$ in the first term of Eq. (B3). The needed term of the form $\delta_{i_1 i_3}$ in the factor $\partial^2 F(a)/\partial a \partial a$ appears in every differentiation of the block of F that contains the contraction of the indices of derivatives:

$$\frac{\partial^2 \Phi_{kk}^{bc}}{\partial a_{i_1 i_2} \partial a_{i_3 i_4}} = \delta_{i_1 i_3} (\delta_{i_2 b} \delta_{i_4 c} + \delta_{i_2 c} \delta_{i_4 b}). \quad (\text{B5})$$

Substituting Eq. (B5) into Eq. (B1) gives the contribution

$$\frac{gC_d(\mu/m)^\varepsilon}{2\varepsilon} \partial_{j_1} \theta_b \partial_{j_1} \theta_c = \frac{gC_d(\mu/m)^\varepsilon}{2\varepsilon} \Phi_{j_1 j_1}^{bc},$$

that is, the block Φ_{kk}^{bc} is reproduced, and the counterterm to the operator F_α in the order $O(1)$ is proportional to the same

monomial F_α . The number of such contributions equals to the number \tilde{n}_α of the contractions between the derivatives in the monomial F_α , and we obtain

$$A_{\alpha\beta} = \frac{1}{2} \tilde{n}_\alpha \delta_{\alpha\beta} + O(1/d), \quad (\text{B6a})$$

$$\gamma_{\alpha\beta}^{(0)}(g_*) = -\tilde{n}_\alpha \delta_{\alpha\beta} + O(1/d), \quad (\text{B6b})$$

$$\Delta_\alpha^{(0)} = (n - \tilde{n}_\alpha) \varepsilon + O(1/d), \quad (\text{B6c})$$

where $0 \leq \tilde{n}_\alpha \leq n$ for operators from the family $(\partial\theta)^{2n}$. The minimal possible value, $\Delta_\alpha = 0$, is reached for $\tilde{n}_\alpha = n$, that is, for the operators where all the derivatives are contracted only with each other. In the family (6.7), one has $\tilde{n}_\alpha = n = 2$ for F_2 and F_5 ; $\tilde{n}_\alpha = 1$ ($\Delta_\alpha = \varepsilon$) for F_3 , F_4 , and F_6 ; $\tilde{n}_\alpha = 0$ ($\Delta_\alpha = 2\varepsilon$) for F_1 .

Let us turn to the calculation of the $O(1/d)$ correction to the results (B6). We write

$$\Delta_\alpha = \Delta_\alpha^{(0)} + C_\alpha/d \quad (\text{B7})$$

with $\Delta_\alpha^{(0)}$ from (B6c) and numerical coefficients C_α determined by the relation

$$\det[\Delta - \Delta_\alpha^{(0)} - C_\alpha/d] = 0, \quad (\text{B8})$$

where Δ is the matrix (4.11) for the family $(\partial\theta)^{2n}$ with a given n .

Since in the $O(1)$ approximation the matrix Δ is diagonal with the diagonal elements $\Delta_\alpha^{(0)}$, all the nondiagonal elements of the matrix in Eq. (B8) are of order $O(1/d)$, as well as its diagonal elements that correspond to the (degenerate) eigenvalue $\Delta_\alpha^{(0)}$. The diagonal elements that correspond to the eigenvalues different from $\Delta_\alpha^{(0)}$ are of order $O(1)$. It then follows that the determinant in Eq. (B8) is of order $O(1/d^N)$, where N is the degeneracy of the eigenvalue $\Delta_\alpha^{(0)}$, and the vanishing of the full determinant in the leading approximation is equivalent to the vanishing of its $N \times N$ subdeterminant that corresponds to $\Delta_\alpha^{(0)}$.

This means that in the $O(1/d)$ approximation, the equations for the coefficients C_α corresponding to different values of $\Delta_\alpha^{(0)}$ or, equivalently, \tilde{n}_α , are independent, and these coefficients can be sought separately.

It is clear from Eq. (B6c) that for small $1/d$, dangerous operators ($\Delta_\alpha < 0$) can be present only among the operators with $\Delta_\alpha^{(0)} = 0$, and below we confine ourselves to this family.

For such operators, $\tilde{n}_\alpha = n$ (tensor indices of the derivatives are contracted only with each other, and the same holds for the indices of the fields), and they always can be represented in the form

$$F = (\phi_1)^{n_1} (\phi_2)^{n_2} \dots (\phi_q)^{n_q}, \quad (\text{B9})$$

where $\sum_{k=1}^q k n_k = n$ and ϕ_k is the scalar operator that contains $2k$ fields θ and cannot be represented as a product. This operator necessarily reduces to the form [cf. Eq. (B4) for $k=1$]

$$\phi_k = \Phi_{s_k s_1 s_1 s_2 s_2 s_3 \dots s_{k-1} s_k}^{l_1 l_1 l_2 l_2 l_3 l_3 \dots l_k l_k} \equiv [s_k s_1, s_1 s_2, s_2 s_3, \dots, s_{k-1} s_k]. \quad (\text{B10})$$

Now let us collect all possible contributions of order $O(1/d)$.

(i) As already mentioned above, an $O(1/d)$ contribution appears from the term with $d_{0,4}$ in Eq. (B3), when the both derivatives in the vertex (6.12) act on ϕ_1 . In Eq. (B1), this gives the contribution of the form

$$\frac{g C_d}{2\varepsilon d} \left(\frac{\mu}{m} \right)^\varepsilon \partial_{j_1} \theta_{j_2} \partial_{j_1} \theta_{j_2} = \frac{g C_d}{2\varepsilon d} \left(\frac{\mu}{m} \right)^\varepsilon \phi_1,$$

that is, the operator F in Eq. (B9) reproduces itself, and the number of such terms is equal to n_1 . Therefore, the corresponding contributions to the matrices $A_{\alpha\beta}$ and $\gamma_{\alpha\beta}^*$ in Eq. (B6) are diagonal and have the forms

$$\delta_1 A_{\alpha\beta} = \frac{n_1}{2d} \delta_{\alpha\beta}, \quad \delta_1 \gamma_{\alpha\beta}^* = -\frac{n_1}{d} \delta_{\alpha\beta}. \quad (\text{B11})$$

(ii) An $O(1/d)$ contribution appears from the term with $d_{0,2}$ in Eq. (B3), when the both derivatives in the vertex (6.12) act on any one of the n factors Φ_{kk}^{bc} in F and produce the delta symbol $\delta_{i_1 i_3}$. Each of these differentiations gives into Eq. (B1) the contribution

$$-\frac{g C_d}{2\varepsilon d} \left(\frac{\mu}{m} \right)^\varepsilon 3 \partial_{j_1} \theta_b \partial_{j_1} \theta_c,$$

and the corresponding contributions into the functions (B6) are again diagonal:

$$\delta_2 A_{\alpha\beta} = -\frac{3n}{2d} \delta_{\alpha\beta}, \quad \delta_1 \gamma_{\alpha\beta}^* = \frac{3n}{d} \delta_{\alpha\beta}. \quad (\text{B12})$$

(iii) The contributions into Eq. (B1) from the first term of Eq. (B3),

$$\frac{g C_d}{4\varepsilon} \left(\frac{\mu}{m} \right)^\varepsilon \frac{\partial^2 F(a)}{\partial a_{i_1 i_2} \partial a_{i_1 i_4}} \partial_{j_1} \theta_{i_2} \partial_{j_1} \theta_{i_4}. \quad (\text{B13})$$

The operation $\partial_{j_1} \theta_{i_2} \partial / \partial a_{i_1 i_2}$ breaks the chain of contractions in ϕ_k ,

$$\partial_{j_1} \theta_{i_2} \partial \phi_k / \partial a_{i_1 i_2} = 2k [j_1 s_1, s_1 s_2, s_2 s_3, \dots, s_{k-1} i_1] \quad (\text{B14})$$

in the notation of Eq. (B10). In the following, the field with the index j_1 is not differentiated, since it does not belong to the operator $F(a)$ in the vertex (6.12).

The operation $\partial_{j_1} \theta_{i_4} \partial / \partial a_{i_1 i_4}$ acts either onto the factor (B14) or onto some other factor ϕ_s in the operator (B9). In the latter case, another broken chain of the form (B14) appears; along with the first broken chain it gives rise to the unbroken chain with $k+s$ elements, that is, ϕ_{k+s} . Therefore, this process gives rise to the counterterm

$$\frac{gC_d}{2\varepsilon d} \left(\frac{\mu}{m}\right)^\varepsilon 4ks\phi_{k+s} \quad (\text{B15})$$

for any pair of factors ϕ_k, ϕ_s in the operator (B9), and they determine nondiagonal contributions to the matrices (B6).

The action of the operation $\partial_{j_1} \theta_{i_4} \partial / \partial a_{i_1 i_4}$ onto the chain (B14) produces a number of different terms. One possibility is the breakdown of the chain of contractions of two kinds:

$$[j_1 s_1, s_1 s_2, \dots, s_{p-1} j_1, i_1 s_{p+1}, \dots, s_{k-1} i_1], \quad (\text{B16a})$$

$$[j_1 s_1, s_1 s_2, \dots, s_{p-1} i_1, j_1 s_{p+1}, \dots, s_{k-1} i_1]. \quad (\text{B16b})$$

The first variant is obviously $\phi_p \phi_{k-p}$, that is, the ‘‘decay’’

$$\phi_{k \rightarrow} \frac{gC_d}{2\varepsilon d} \left(\frac{\mu}{m}\right)^{\varepsilon k-1} \sum_{p=1}^{k-1} k \phi_p \phi_{k-p}. \quad (\text{B17})$$

The second variant gives $(k-1)$ factors ϕ_k , and they give the diagonal contribution to the matrices (B6):

$$\delta_3 A_{\alpha\beta} = \frac{1}{2d} \sum_{k=2}^q n_k k(k-1) \delta_{\alpha\beta} \quad (\text{B18})$$

with n_k and q from Eq. (B9).

Finally, the differentiation of the rightmost factor in Eq. (B14) gives $d\phi_k$. This is the leading $O(1)$ contribution, but it also gives the $O(1/d)$ term after the substitution

$$g \rightarrow g_* = \frac{2\varepsilon}{C_d} \left(1 + \frac{2}{d}\right) + O(1/d^2).$$

The contribution to the matrices (B6) is also diagonal:

$$\delta_4 A_{\alpha\beta} = \frac{n}{d} \delta_{\alpha\beta}, \quad \delta_4 \gamma_{\alpha\beta}^* = -\frac{2n}{d} \delta_{\alpha\beta}. \quad (\text{B19})$$

Collecting the contributions (B11), (B12), (B18), and (B19) gives

$$\delta A_{\alpha\beta} = \frac{1}{2d} \left[n_1 - n + \sum_{k=2}^q n_k k(k-1) \right] \delta_{\alpha\beta}, \quad (\text{B20a})$$

$$\delta \gamma_{\alpha\beta}^* = \frac{1}{d} \left[n - n_1 - \sum_{k=2}^q n_k k(k-1) \right] \delta_{\alpha\beta} \quad (\text{B20b})$$

for the total diagonal $O(1/d)$ contribution to the matrices (B6), while their nontrivial nondiagonal elements are determined by Eqs. (B15) and (B17), with the summation in the former over all pairs of the factors $\phi_k \phi_s$ in the operator (B9).

Consider a few examples that illustrate the above algorithm and lead to the results announced in Sec. VI C.

For $n=2$, the family of operators with $\tilde{n}_\alpha = n=2$ consists of two elements,

$$F = \{\phi_1^2, \phi_2\},$$

and the equation (B8) for the coefficients C_α in the representation (B7) has the form (here and below we omit the subscript α and change the signs of the matrix elements so that they all become positive):

$$\begin{vmatrix} C & 4 \\ 2 & C \end{vmatrix} = 0. \quad (\text{B21})$$

The solutions are $C = \pm 2\sqrt{2}$, as announced in Eq. (6.27a).

For $n=3$, the relevant family consists of three elements,

$$F = \{\phi_1^3, \phi_1 \phi_2, \phi_3\},$$

the equation has the form

$$\begin{vmatrix} C & 12 & 0 \\ 2 & C & 8 \\ 0 & 6 & 3+C \end{vmatrix} = 0 \quad (\text{B22})$$

with the solutions given in Eq. (6.27b).

For $n=4$, the relevant family consists of five elements,

$$F = \{\phi_1^4, \phi_1^2 \phi_2, \phi_2^2, \phi_1 \phi_3, \phi_4\},$$

the equation has the form

$$\begin{vmatrix} C & 24 & 0 & 0 & 0 \\ 2 & C & 4 & 16 & 0 \\ 0 & 4 & C & 0 & 16 \\ 0 & 6 & 0 & 3+C & 12 \\ 0 & 0 & 4 & 8 & 8+C \end{vmatrix} = 0 \quad (\text{B23})$$

with the solutions given in Eq. (6.27c).

For $n=5$, the relevant family consists of seven elements,

$$F = \{\phi_1^5, \phi_1^3 \phi_2, \phi_1 \phi_2^2, \phi_1^2 \phi_3, \phi_1 \phi_4, \phi_2 \phi_3, \phi_5\},$$

the equation has the form

$$\begin{vmatrix} C & 40 & 0 & 0 & 0 & 0 & 0 \\ 2 & C & 12 & 24 & 0 & 0 & 0 \\ 0 & 4 & C & 0 & 16 & 16 & 0 \\ 0 & 6 & 0 & 3+C & 24 & 4 & 0 \\ 0 & 0 & 4 & 8 & 8+C & 0 & 16 \\ 0 & 0 & 6 & 2 & 0 & 3+C & 24 \\ 0 & 0 & 0 & 0 & 10 & 10 & 15+C \end{vmatrix} = 0 \quad (\text{B24})$$

with the solutions given in Eq. (6.27d).

For $n=6$, the relevant family consists of eleven elements,

$$F = \{\phi_1^6, \phi_1^4 \phi_2, \phi_1^2 \phi_2^2, \phi_2^3, \phi_1^3 \phi_3, \phi_1 \phi_2 \phi_3, \phi_3^2, \phi_1^2 \phi_4, \phi_2 \phi_4, \phi_1 \phi_5, \phi_6\},$$

the equation has the form

$$\begin{array}{cccccccccccc|l}
C & 60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
2 & C & 24 & 0 & 32 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 4 & C & 4 & 0 & 32 & 0 & 16 & 0 & 0 & 0 & 0 & \\
0 & 0 & 6 & C & 0 & 0 & 0 & 0 & 48 & 0 & 0 & 0 & \\
0 & 6 & 0 & 0 & 3+C & 12 & 0 & 36 & 0 & 0 & 0 & 0 & \\
0 & 0 & 6 & 0 & 2 & 3+C & 8 & 0 & 12 & 24 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 12 & 6+C & 0 & 0 & 0 & 0 & 36 & \\
0 & 0 & 4 & 0 & 8 & 0 & 0 & 8+C & 4 & 32 & 0 & 0 & \\
0 & 0 & 0 & 4 & 0 & 8 & 0 & 2 & 8+C & 0 & 32 & 0 & \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 10 & 0 & 15+C & 20 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 12 & 12 & 24+C & 0 & \\
\hline
& & & & & & & & & & & & =0 & (B25)
\end{array}$$

with the solutions given in Eq. (6.27e).

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